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Orbit Lengths and Character Degrees in p-Sylow Subgroups of Some Classical Lie Groups

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1. INTRODUCTION

Let $\mathfrak{G}$ denote $SL(n, q)$ or $Sp(2n, q)$, the linear or symplectic group over $\mathbb{F}_q$, q a p-power. Let $\mathfrak{P}$ be a p-Sylow subgroup of $\mathfrak{G}$. We determine the set of conjugacy class lengths of suitable normal abelian subgroups $\mathfrak{U}$ of $\mathfrak{P}$ and use this information to get the character degrees of $\mathfrak{P}$. Given a group G acting on a vector space V, it is known that V and its dual $\hat{V}$ are not necessarily G-equivalent, if the action is not coprime. We introduce a weaker version of G-equivalence and show that this is enough to assure equality between the set of conjugacy class lengths and character degrees when $q = p$, p odd. Some partial results are obtained when $q = p^m$, $m > 1$, using a linearization technique involving formal power series. Some results have been determined by Huppert (see [6]) for $SL(n, q)$. With a slightly different approach, we will obtain them *ex novo* and extend them to the symplectic case. The proof goes along the following lines:

(a) Determine a big abelian normal subgroup $\mathfrak{U}$ of $\mathfrak{P}$ such that

$$1 \to \mathfrak{U} \xrightarrow{j} \mathfrak{P} \xrightarrow{\pi} \tilde{\mathfrak{T}} \to 1 \tag{1}$$

is a splitting exact sequence.

(b) Determine the orbit lengths of the action of $\mathfrak{T}$ on $\mathfrak{U}$, where $\mathfrak{T} \cong \tilde{\mathfrak{T}}$ is the complement of $\mathfrak{U}$ in $\mathfrak{P}$.

(c) Prove that the same lengths appear in the dual action.

(d) Use some results of Clifford's theory to get the desired character degrees when $q = p$.

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2. NOTATIONS AND PRELIMINARY RESULTS

Given a group G, we denote with $Irr(G)$ the set of its irreducible characters. As standard G_p will denote a p-Sylow subgroup of G. Suppose $\chi \in Irr(N)$, where $N \trianglelefteq G$, then $Irr(N)$ is a G-set via $\chi^g(x^g) = \chi(x)$, $g \in G$, $x \in N$. Call $I_G(\chi) = C_G(\chi)$ the inertia subgroup of χ. Set $cdG = \{\chi(1) : \chi \in Irr(G)\}$. Suppose Ω is a G-set, then $\omega^G = \{\omega^g : g \in G\}$ is the G-orbit of $\omega \in \Omega$ and $\mathcal{O}(\Omega) = \{|\omega^G| : \omega \in \Omega\}$.

DEFINITION 2.1. Given two G-sets Ω_1, Ω_2, we say they are G-equivalent if there exists a bijection σ from Ω_1 onto Ω_2 such that $(\omega_1^g)^\sigma = (\omega_1^\sigma)^g$ for every $g \in G$ and weak equivalent if $\mathcal{O}(\Omega_1) = \mathcal{O}(\Omega_2)$.

Usually gothic letters will be used for groups of Lie type and capital ones for matrices. Given a matrix T, $T^* = (T^{-1})^t$ will denote the adjoint matrix. We denote with E_{ij} the elementary matrix with entry 1 in the position (i, j) and zero elsewhere.

LEMMA 2.2. *Suppose G is a splitting extension of its normal subgroup N, then any linear character $\lambda \in Irr(N)$ can be extended to its inertia subgroup $I_G(\lambda)$.*

Proof. Suppose K is a complement of N in G, then, by Dedekind's identity, is $I = I_G(\lambda) = N(I_G(\lambda) \cap K) = N\tilde{K}$. Define $\lambda_0(ax) := \lambda(a)$, where $a \in N$, $x \in \tilde{K}$. Let $ab = a'b'$, then $a^{-1}a' = bb'^{-1} \in N \cap \tilde{K} = 1$, so $a = a'$, $b = b'$, and λ_0 is well defined.

$$\lambda_0(axby) = \lambda_0\left(ab^{x^{-1}}xy\right) = \lambda(ab) = \lambda(a)\lambda(b) = \lambda_0(ax)\lambda_0(by) \quad (2)$$

proves that also λ_0 is a character of $I_G(\lambda)$ extending λ (compare also [9, Ex. 6.18] for a more general assertion). ∎

LEMMA 2.3. *Suppose $G = A \rtimes T$, where $A \trianglelefteq G$ abelian, then $cdG = \{\beta(1)|G : I_G(\lambda)| : \lambda \in Irr(A)$ and $\beta \in Irr(I_G(\lambda)/A)\}$.*

Proof. Let $\psi \in Irr(G)$, then $\psi_A = e(\psi)\Sigma_i \lambda_i$, hence, by Frobenius' reciprocity law, $e(\psi) = (\psi_A, \lambda) = (\psi, \lambda^G)$. But, by Gallagher's theorem (see [9, p. 85]), $\lambda^G = \Sigma_i(\lambda_0\beta_i)^G$, where $\lambda_0 \in Irr(I_G(\lambda))$ is a linear character extending λ and β_i varies in $Irr(I_G(\lambda)/A)$. By Clifford's theory $(\lambda_0\beta_i)^G \in Irr(G)$, then $\psi = (\lambda_0\beta_i)^G$ for a suitable index i. But, as easy consequence of the definition of induced character,

$$(\lambda_0\beta_i)^G(1) = \lambda_0(1)\beta_i(1)|G : I_G(\lambda)| = \beta_i(1)|G : I_G(\lambda)|, \quad (3)$$

as asserted. ∎

In particular, for β, the trivial character in $Irr(I_G(\lambda)/A)$, $\lambda_0^G \in Irr(G)$, so $\lambda_0^G(1) = |G : I_G(\lambda_0)| \in cdG$. Our problem reduces to the search of suitable linear characters of a normal subgroup, in such a way that any character degree could be obtained as above described. It has been conjectured that for $\mathfrak{P} \in Syl_p(\mathscr{L}_n(\mathbb{F}_q))$, $\mathscr{L}_n(\mathbb{F}_q)$ a classical Lie group, $cd\mathfrak{P}$ is a set of q-powers. By the preceding lemma this would imply that $|G : I_G(\lambda_0)|$ is a q-power. We will prove that this holds in our setting, providing some weak evidence for this conjecture. Remark that the conjecture on character degrees is equivalent to state that $\forall A \trianglelefteq G$, $\forall \lambda \in Irr(A)$ and $\forall \beta \in Irr(I_G(\lambda)/A)$, $\beta(1)$ is a q-power. Observe that in the symplectic situation $I_G(\lambda)/A \lesssim \mathfrak{T}_1 \times \mathfrak{T}_2$, where $\mathfrak{T}_i \in Syl_p(SL(m_i, q))$ for suitable m_i. Since any finite p-group can be embedded in such groups, the truth of the character degree conjecture would imply a strong restriction on the structure of the inertia subgroups $I_{\mathfrak{P}}(\lambda)$. We now carry over the proof distinguishing the two possible cases.

3. ORBIT LENGTHS

In [6], Huppert defined the following function:

$$f(n) = \begin{cases} \dfrac{(n-1)^2}{4}, & n \text{ odd}, \\ \dfrac{n(n-2)}{4}, & n \text{ even}. \end{cases} \tag{4}$$

The main result of this and the next section is:

THEOREM 3.4 (Orbit Lengths). *Suppose $\mathfrak{P} \in Syl_p(\mathscr{L}_n(\mathbb{F}_q))$, where $\mathscr{L}_n(\mathbb{F}_q)$ is $SL(n, q)$ or $Sp(n, q)$, q odd in the symplectic case. Set*

$$g(n) = \begin{cases} f(n), & \mathscr{L}_n(\mathbb{F}_q) \text{ linear}, \\ \begin{pmatrix} \frac{n}{2} \\ 2 \end{pmatrix}, & \mathscr{L}_n(\mathbb{F}_q) \text{ symplectic}, \end{cases} \tag{5}$$

then $\mathscr{O}(\mathfrak{U}) = \{q^j : 0 \leq j \leq g(n)\}$.

We will need this well known lemma:

LEMMA 3.5. *A is a maximal abelian subgroup of G iff A is self-centralizing.*

Proof. Compare [5]. ∎

3.1. *The Linear Case*

THEOREM 3.6. *$\mathfrak{P} \in Syl_p(SL(n, q))$ is a splitting extension of a maximal abelian normal subgroup $\mathfrak{U}$.*

Proof. Set $k = n - r$ and consider

$$\mathfrak{U} = \left\{ \begin{pmatrix} I_r & \mathbf{0} \\ A & I_k \end{pmatrix} : A \in (\mathbb{F}_q)_{k \times r} \right\} \tag{6}$$

and

$$\mathfrak{T}_{rk} = \mathfrak{T} = \left\{ \begin{pmatrix} X & \mathbf{0} \\ \mathbf{0} & Y \end{pmatrix} : X \in SL(r, \mathbb{F}_q)_p, Y \in SL(k, \mathbb{F}_q)_p \right\}; \tag{7}$$

then $\mathfrak{P} = \mathfrak{U}\mathfrak{T}$, $\mathfrak{U} \cap \mathfrak{T} = 1$, and $\mathfrak{U}$ is an elementary abelian normal subgroup (it is $\mathfrak{T}$-invariant).

We now show that $\mathfrak{U}$ is maximal in $\mathscr{A} = \{\mathfrak{C} \leq \mathfrak{P} : \mathfrak{C}' = 1\}$. It's enough to prove that $\mathfrak{U}$ self-centralizes. If not, by Dedekind's identity, there is an element of $\mathfrak{T}$ commuting with every element of $\mathfrak{U}$:

$$\begin{pmatrix} X^{-1} & \mathbf{0} \\ \mathbf{0} & Y^{-1} \end{pmatrix} \begin{pmatrix} I & \mathbf{0} \\ A & I \end{pmatrix} \begin{pmatrix} X & \mathbf{0} \\ \mathbf{0} & Y \end{pmatrix} = \begin{pmatrix} I & \mathbf{0} \\ Y^{-1}AX & I \end{pmatrix}. \tag{8}$$

Hence $Y^{-1}AX = A$, $\forall A \in (\mathbb{F}_q)_{k \times r}$. Bordering A and X as $A = (A_1, \mathbf{a}'_1)$, $X = \begin{pmatrix} X_1 & \mathbf{0} \\ \mathbf{x}_2 & 1 \end{pmatrix}$, we get

$$\left(Y^{-1}A_1X_1 + Y^{-1}\mathbf{a}'_1\mathbf{x}_2, Y^{-1}\mathbf{a}'_1\right) = \left(A_1, \mathbf{a}'_1\right). \tag{9}$$

Since $\mathbf{a}'_1$ is arbitrary, this implies that $Y = I$. Setting $A_1 = I, \mathbf{a}_1 \neq \mathbf{0}$, we get $X_1 = I$. Finally, for $A_1 = \mathbf{0}$, we get $\mathbf{x}_2 = \mathbf{0}$. ∎

We are in particular interested in those normal abelian subgroups of minimal index. This corresponds to the following choice for r.

Set $\varepsilon \equiv n \pmod 2$, where $\varepsilon \in \{0, 1\}$ and $r = (n + \varepsilon)/2$. The importance of establishing that $\mathfrak{U} \in \mathscr{A}$, is explained by a theorem of Itô stating that the degree of irreducible characters of a group has to divide the index of subnormal abelian subgroups. We now carry out the second step, i.e., determine the orbit lengths of $\mathfrak{U}$ considered as a $\mathfrak{T}$-set. As will be clear at the end of the proof, we could at once attack the problem in the dual space $\hat{\mathfrak{U}}$, but our approach reflects the general philosophy of considering first the direct action (this being generally easier) and from this retrieving information on the dual action. Much is known (see [1, 4, 8]) when the action is coprime, that is, when T is a group acting on a group A, and $(|A|, |T|) = 1$. The purpose of this approach is also to find a way to generalize such results when the action is no more coprime. The problem is that A and $Irr(A)$ need not be equivalent T-sets when coprimality drops down (see [7, Vol. II, p. 121]). It would be interesting to understand whether a weaker condition still holds:

Question. Is it true that A and $Irr(A)$, considered as T-sets are weakly equivalent?

We shift back to gothic notations. As we will see, the proof in the linear case is by far the most complex.

THEOREM 3.7. *$\mathcal{O}(\mathfrak{U}) \supseteq \{q^j : 0 \le j \le f(n)\}$ where $q^{f(n)} = |\mathfrak{B} : \mathfrak{U}|$.*

Proof. The second part of the statement is straightforward. The action on $\mathfrak{U}$ is described by $Y^{-1}AX$ as above seen. Since the inversion is an anti-isomorphism of $SL(k, \mathbb{F}_q)_p$, we may substitute Y^{-1} with Y. We distinguish the cases $n = 2k$ and $n = 2k + 1$. We will now proceed by induction on k, showing how the inductive hypothesis at the kth level implies the truth of the assertion for $n = 2k + 1, 2k + 2 = 2(k + 1)$. If $k = 1$, then $n = 1, 2$ and $\mathfrak{B}$ is abelian. By induction, $\exists B \in (\mathbb{F}_q)_{2k}$ which are representatives of the orbit lengths from 1 up to $q^{k(k-1)}$. We refine the induction hypothesis claiming the existence of $B_0, \dots, B_{k-1} \in (\mathbb{F}_q)_{2k}$ such that:

(a) B_i is non-singular.
(b) $|B_i^{\mathfrak{B}}| = q^{k(k-1)-i}$.

We will call such matrices the carriers of long orbits.

Step 1.We first search representatives for short orbits. Just consider this partition:

$$Y(B, \mathbf{b}^t)\begin{pmatrix} T & \mathbf{0} \\ \mathbf{t} & 1 \end{pmatrix} = (YBT + Y\mathbf{b}^t, Y\mathbf{b}^t) \tag{10}$$

and set $\mathbf{b} = \mathbf{0}$. By induction, there are matrices $B \in (\mathbb{F}_q)_k$, with orbit lengths from 1 up to $q^{k(k-1)}$; then $(B, \mathbf{0})$ provide the same lengths.

Step 2. Consider the following situation:

$$Y(\mathbf{b}^t, B)\begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{t}^t & T \end{pmatrix} = \left(Y(\mathbf{b}^t + B\mathbf{t}^t), YBT\right). \tag{11}$$

Here $Y, T \in (\mathbb{F}_q)_k$ are lower triangular unipotent matrices. The idea is to get elements representing long orbits saturating the first column of the above matrix for any choice of Y. Since Y is invertible, this request does not really depend on Y. In fact,

$$\{Y(\mathbf{b}' + B\mathbf{t}') : \mathbf{t} \in \mathbb{F}_q^k\} = \{\mathbf{b}' + B\mathbf{t}' : \mathbf{t} \in \mathbb{F}_q^k\} = \{B\mathbf{t}' : \mathbf{t} \in \mathbb{F}_q^k\} = \mathbb{F}_q^k$$

if and only if B represents a surjective linear operator in $\mathrm{End}(\mathbb{F}_q^k)$, that is, if and only if B is non-singular. But, by induction, there exist k elements B_i which are the carriers of long orbits at the kth level. Set, for example, $\mathbf{b} = \mathbf{0}$, then $(\mathbf{0}, B_i)$ will have orbit length q^{k^2-i}, $i = 0, \dots, k-1$. Observe that those steps provide every orbit length when $n = 2k+1$.

Step 3. To get short orbit representatives, just consider

$$\begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{t}' & \mathbf{t} \end{pmatrix} \begin{pmatrix} \bar{\mathbf{b}} \\ \bar{B} \end{pmatrix} X = \begin{pmatrix} \bar{\mathbf{b}} X \\ \mathbf{t}'\bar{\mathbf{b}}X + T\bar{B}X \end{pmatrix} \tag{12}$$

and set $\bar{\mathbf{b}} = \mathbf{0}$. By the comment at the end of Step 2, we get all lengths from 1 up to q^{k^2}.

Step 4. Now we partition the matrix $Y \in (\mathbb{F}_q)_{k+1}$, hence

$$\begin{pmatrix} T & \mathbf{0} \\ \mathbf{t} & 1 \end{pmatrix} \begin{pmatrix} \bar{B} \\ \bar{\mathbf{b}} \end{pmatrix} X = \begin{pmatrix} T\bar{B}X \\ (\mathbf{t}\bar{B} + \bar{\mathbf{b}})X \end{pmatrix}. \tag{13}$$

Again, saturation of the last row, for any X, is implied as soon as $\bar{B}$ is of maximal rank, that is, $rk\bar{B} = k$. That surely happens if we set $\bar{B} = (\mathbf{0}\ B)$, where B is a long orbit carrier in step 1. So the elements $\binom{\bar{B}_i}{\bar{\mathbf{b}}}$ are long orbit representatives for any choice of $\bar{\mathbf{b}}$. If we set $\bar{\mathbf{b}} = \mathbf{e}_1$, then these are also non-singular, hence long orbit carriers. Observe that there are k, with orbit length $q^{k(k+1)-i}$, $i = 0, \dots, k-1$.

Step 5. Unfortunately Step 3 provides a singular element with orbit length q^{k^2}. To get a non-singular one, take the long orbit carrier B_0 and embed it in the non-singular matrix $\begin{pmatrix} B_0 & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}$. Then

$$\begin{pmatrix} T & \mathbf{0} \\ \mathbf{t} & 1 \end{pmatrix} \begin{pmatrix} B_0 & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} S & \mathbf{0} \\ \mathbf{s} & 1 \end{pmatrix} = \begin{pmatrix} TB_0S & \mathbf{0} \\ \mathbf{t}B_0S + \mathbf{s} & 1 \end{pmatrix}. \tag{14}$$

Since $\mathbf{s}$ varies arbitrarily in $\mathbb{F}_q^k$, this matrix has orbit length $q^{k(k-1)+k}$. ∎

Remark 3.1. The precedent proof also shows how to construct elements with a prescribed orbit length. Let us just work out the calculation when

$k = 1$. Take the matrix (1). Bordering it as in Steps 1 and 3, we have $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, which is central. Bordering it as in Steps 2 and 4, we get $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ with orbit length q^2. Using Step 5, we get $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ with orbit length q.

Remark 3.2. It is proved once more than $\mathfrak{U}$ is a maximal abelian subgroup. In fact there are elements of maximal possible orbit length $|A^{\mathfrak{T}}| = |\mathfrak{T}|$, which is equivalent to $C_{\mathfrak{T}}(A) = 1$ and, by Dedekind's identity, one may conclude that $\mathfrak{U} = C_{\mathfrak{P}}(\mathfrak{U})$.

3.2. *The Symplectic Case*

Once the theme has been settled, let us just play the variations.

THEOREM 3.8. *$\mathfrak{P} \in Syl_p(Sp(2n, q))$ is a splitting extension of a maximal abelian normal subgroup $\mathfrak{U}$.*

Proof. We first investigate the structure of $\mathfrak{P}$. As is well known, there exists, up to equivalence, just one symplectic form which can be described by the matrix

$$E = \begin{pmatrix} \mathbf{0} & I_n \\ -I_n & \mathbf{0} \end{pmatrix}. \tag{15}$$

Consider now the equation

$$\begin{pmatrix} T^t & B^t \\ \mathbf{0} & C^t \end{pmatrix} \begin{pmatrix} \mathbf{0} & I_n \\ -I_n & \mathbf{0} \end{pmatrix} \begin{pmatrix} T & \mathbf{0} \\ B & C \end{pmatrix} = \begin{pmatrix} \mathbf{0} & I_n \\ -I_n & \mathbf{0} \end{pmatrix}; \tag{16}$$

that is, $C = T^{-t}$ and $B = AT$, where $A^t = A$. Consider now the decomposition

$$\begin{pmatrix} T & \mathbf{0} \\ AT & T^{-t} \end{pmatrix} = \begin{pmatrix} I & \mathbf{0} \\ A & I \end{pmatrix} \begin{pmatrix} T & \mathbf{0} \\ \mathbf{0} & T^{-t} \end{pmatrix}. \tag{17}$$

$\mathfrak{T}$ acts on $\mathfrak{U}$ via $A \to T^tAT$; in fact,

$$\begin{pmatrix} T^{-1} & \mathbf{0} \\ \mathbf{0} & T^t \end{pmatrix} \begin{pmatrix} I & \mathbf{0} \\ A & I \end{pmatrix} \begin{pmatrix} T & \mathbf{0} \\ \mathbf{0} & T^{-t} \end{pmatrix} = \begin{pmatrix} I & \mathbf{0} \\ T^tAT & I \end{pmatrix}. \tag{18}$$

This proves that $\mathfrak{P} = \mathfrak{T} \ltimes \mathfrak{U}$, where $\mathfrak{U}$, $\mathfrak{T}$ are the groups constituted by the matrices which appear as the first and the second factor, respectively, in (17). To prove that $\mathfrak{P}$ is a p-Sylow subgroup, just consider that its order is $q^{\binom{n}{2}+\binom{n+1}{2}}$; but that is exactly the maximal p-power dividing the group order (see [2, Chap. I] or [3]). As in the linear case, to prove the maximality of $\mathfrak{U}$, it is enough to show that $T = I$, whenever $T \in C_{\mathfrak{T}}(\mathfrak{U})$. Set $A = I$, then $T'T = I$. So $T = (T')^{-1} = T^*$. But T is lower triangular with 1 along the diagonal; hence, $T = I$. ▮

THEOREM 3.9. $\mathscr{O}(\mathfrak{U}) \supseteq \{q^j : 0 \le j \le \binom{n}{2}\}$.

Proof. Observe that $q^{\binom{n}{2}} = |\mathfrak{B} : \mathfrak{U}|$. Now proceed by induction on n. If $n = 1$ there is nothing to show. Let $n > 1$ and partition A and T in four submatrices. Then we have

$$\begin{pmatrix} T' & \mathbf{t}' \\ \mathbf{0} & 1 \end{pmatrix}\begin{pmatrix} B & \mathbf{b}' \\ \mathbf{b} & \beta \end{pmatrix}\begin{pmatrix} T & \mathbf{0} \\ \mathbf{t} & 1 \end{pmatrix}$$
$$= \begin{pmatrix} T'BT + \mathbf{t}'\mathbf{b}T + T'\mathbf{b}'\mathbf{t} + \mathbf{t}'\mathbf{t}\beta & T'\mathbf{b}' + \beta\mathbf{t}' \\ \mathbf{b}T + \beta\mathbf{t} & \beta \end{pmatrix}. \tag{19}$$

Step 1. By induction there are matrices $B_i \in \mathfrak{U}_{n-1} : q^i = |B_i^{\mathfrak{T}_{n-1}}| = |B_i^{\mathfrak{T}_n}|$, where B_i can be thought of as embedded in $\mathfrak{U}_n$ setting $\mathbf{b} = \mathbf{0}$, $\beta = 0$.

Step 2. Set $B = \mathbf{0}$, $\mathbf{b} = \mathbf{0}$, $\beta = 1$; then $\left(\begin{smallmatrix} \mathbf{t}'\mathbf{t} & \mathbf{t}' \\ \mathbf{t} & 1 \end{smallmatrix}\right)$ generates an orbit of length q^{n-1}.

By Step 1, $\{q^j : 0 \le j \le \binom{n-1}{2}\} \subseteq \mathscr{O}(\mathfrak{U})$.

By Step 2, $\{q^j : n - 1 \le j \le \binom{n}{2}\} \subseteq \mathscr{O}(\mathfrak{U})$, since $\left(\begin{smallmatrix} T'A_iT & \mathbf{0} \\ \mathbf{0} & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \mathbf{t}'\mathbf{t} & \mathbf{t}' \\ \mathbf{t} & 1 \end{smallmatrix}\right)$ vary independently. By the first observation, $\mathscr{O}(\mathscr{U})$ cannot contain a greater q-power. ▮

Remark 3.3. We may here provide exactly a set of elements each for any orbit length. A careful analysis of the proof shows that only diagonal elements are needed. In fact, $\mathbf{b} = \mathbf{0}$ in both cases. Consider now

$$A(i,r) = \begin{pmatrix} I_i & & & & & \\ & 0 & & & & \\ & & I_{r-i-1} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}; \tag{20}$$

that is,

$$A(i,r) = \operatorname{diag}\Big(\underbrace{1,\dots,1}_{i}, 0, \overbrace{1,\dots,1}^{r-i-1}, 0, \dots, 0\Big).$$

Then, an easy induction argument and the proof of the precedent theorem show that $A(i, r)$ has an orbit of length $q^{\binom{r}{2}-i}$. Since $A(r-1, r) = A(0, r-1)$, set, for uniqueness, the restriction $0 \le i \le r-2$, where $1 \le r \le n$.

4. LIE METHODS

In this section, we will prove the reverse inclusion, that is, any orbit length is a q-power, unless $\mathfrak{B}$ symplectic and $p = 2$. The idea is to extend some of the methods used by Chevalley to define the finite correspondent of automorphism groups of simple Lie algebras. In his considerations a central role is played by the exponential function defined over a nilpotent $\mathbb{C}$-algebra of operators. The principal obstruction in this approach is that this map cannot be defined for $\mathbb{F}_q$-algebras when the index of nilpotency overcomes p, the characteristic of the finite field. We will prove there exists a family of maps providing bijections between particular nilpotent $\mathbb{F}_q$-algebras and linear groups. We will find a particularly simple map for $\mathfrak{B} = SL(n, q)_p$ which will allow us to prove that every conjugacy class in $\mathfrak{B}$ has q-power order.

DEFINITION 4.10. Denote with $K[[x]]$ the K-algebra of formal power series over the field K.

DEFINITION 4.11. Denote with $\mathcal{N}_n$ the nilpotent K-algebra of strictly lower triangular matrices of $(K)_n$.

DEFINITION 4.12. Suppose $\mathcal{N}$ is a nilpotent algebra, denote with $\nu(\mathcal{N})$ the index of nilpotency of $\mathcal{N}$, that is, the minimal integer such that $N^\nu = 0$, $\forall N \in \mathcal{N}$.

In particular, we have $\nu(\mathcal{N}_n) = n$.

LEMMA 4.13. *Suppose M_1, M_2 are elements of $\mathcal{N}_k$, $\mathcal{N}_r$, respectively. Let $f = \sum_i f_i x^i \in K[[x]]$ and $A \in (K)_{k\times r}$. Set $Y = M_1 A - AM_2$; then*

$$f(M_1)A = Af(M_2) + \sum_{i=1}^{\infty} \sum_{j=0}^{i-1} f_i M_1^j Y M_2^{i-1-j}.$$

Proof. We first prove by induction on i that

$$M_1^i A = AM_2^i + \sum_{j=0}^{i-1} M_1^j Y M_2^{i-1-j}. \tag{21}$$

For $i = 1$, we have $\sum_{j=0}^{i-1} M_1^j Y M_2^{i-1-j} = Y$, which is just the definition of Y. Now

$$M_1^{i+1}A = M_1\left(AM_2^i + \sum_{j=0}^{i-1} M_1^j Y M_2^{i-1-j}\right)$$
$$= AM_2^{i+1} + YM_2^i + \sum_{j=0}^{i-1} M_1^{j+1} Y M_2^{i-(j+1)}$$
$$= AM_2^{i+1} + \sum_{j=0}^{i} M_1^j Y M_2^{i-j}, \qquad (22)$$

completing the induction step. Multiply now by f_i and sum up. Observe that the number of summands is actually finite by the nilpotency of M_j. ∎

We now need to establish some facts about nilpotent commutative algebras.

LEMMA 4.14. *Suppose x, y are nilpotent operators and $xy = yx$. Set $\nu(x) = i$ and $\nu(y) = j$, then $(x + y)^{i+j-1} = 0$; in particular, $x + y$ is nilpotent.*

Proof. Use the binomial theorem and observe that $x^n y^m = 0$ whenever $n + m \geq i + j - 1$. ∎

DEFINITION 4.15. Set $\Phi_{M_1,M_2}(Y) = \sum_{i=1}^{\infty} \sum_{j=0}^{i-1} f_i M_1^j Y M_2^{i-1-j}$, where $Y \in (K)_{k \times r}$ and $M_1 \in \mathcal{N}_k$, $M_2 \in \mathcal{N}_r$.

LEMMA 4.16. *If $Y \in (K)_{k\times r}$ and $M_1 \in \mathcal{N}_k$, $M_2 \in \mathcal{N}_r$, then $\Phi_{M_1,M_2}(Y)$ is a linear operator in Y and is invertible if and only if $f_1 \neq 0$.*

Proof. The linearity is obvious. Denote now with $\lambda = \lambda_{M_1}$ the operator $Y \xrightarrow{\lambda} M_1 Y$ and with $\rho = \rho_{M_2}$ the operator $Y \xrightarrow{\rho} YM_2$. Clearly λ and ρ are commuting nilpotent operators. Now

$$\Phi_{M_1,M_2} = \sum_{i\geq 0} f_{i+1} \sum_j \lambda^j \rho^{i-j} = f_1 I + \sigma. \qquad (23)$$

Applying the preceding lemma repeatedly, we see that σ is a nilpotent operator. Hence Φ_{M_1,M_2} is invertible iff $f_1 \neq 0$. ∎

Suppose now that $charK = 0$ or $n \leq charK$, then $\exp(M) = \sum_{i=0}^{n-1} M^i/i!$ is well defined for every element in $\mathcal{N}_n$. Suppose $Y = 0$, then we have the following result:

COROLLARY 4.17. *Suppose $M_1 A + AM_2 = 0$, $M_i \in \mathcal{N}_n$ and $A \in (K)_n$, then* $\exp(M_1)A\exp(M_2) = A$.

Proof. Simply observe that $\exp(-M_2) = \exp(M_2)^{-1}$ and apply Lemma 4.13. ∎

We point out two facts:

(a) We have used the well known property of the classical exponential map $\exp(-x) = \exp(x)^{-1}$ over $\mathbb{C}$. This property carries over even when exp is defined as a formal power series repeating exactly the proof for the complex case.

(b) The last result provides a correspondence between nilpotent operators and elements that, roughly speaking, centralize the matrix A.

The aim is to construct a bijection from $\mathcal{N}_n$ onto $\mathfrak{T}_n$, where $\mathfrak{T}_n$ denotes the unipotent lower triangular matrices in $(K)_n$, through a suitable power series f sharing some of the properties of the exponential map. To attain a simpler notation, we will not distinguish between the power series f and the operator induced by it on $\mathcal{N}_n$.

THEOREM 4.18. *There exists* $f = 1 + x + \sum_{i \geq 2} f_i x^i \in \mathbb{F}_q[[x]]$, $p > 2$, *with the following properties*:

(a) $f(-N) = f(N)^{-1} \ \forall N \in \mathcal{N}_n$.

(b) $f(M_1)Af(-M_2) = A \Leftrightarrow M_1 A = AM_2$.

(c) *f induces a bijection from $\mathcal{N}_n$ onto $\mathfrak{T}_n$.*

Observe that we do not require $\dim M_1 = \dim M_2$.

Proof. Set $f(x) := \sum_{i \geq 0} f_i x^i$. We will show that for $p > 2$ it is possible to determine the coefficients $f_i \in \mathbb{F}_q$ in such a way that the above conditions are satisfied.

Step 1. Given $N \in \mathcal{N}$, clearly $f(N)$ is lower triangular and unipotency is equivalent to $f_0 = 1$.

Step 2. Set $g(x) = f(x)f(-x) = \sum_{l \geq 0} c_l x^l$. Clearly $g(x) = g(-x)$, hence $c_{2l+1} = 0$. We have only to consider the conditions

$$\begin{cases} c_0 = 1 \\ c_l = 0 \qquad 0 < l < n \end{cases}. \tag{24}$$

Now $c_0 = f_0^2 = 1$. Suppose inductively that we have determined f_i for $i < 2k$. The condition $c_{2k} = 0$ is satisfied as soon as

$$2f_{2k} = 2f_{2k-1}f_1 - 2f_{2k-2}f_2 + \cdots + 2(-1)^k f_{k+1}f_{k-1} + (-1)^{k+1}f_k^2, \tag{25}$$

which is solvable in f_{2k} since $p > 2$. Observe that no condition is given on the odd indexed coefficients.

Step 3. By Lemma 4.13, with $Y = 0$, we already still know that $M_1 A = AM_2 \Rightarrow f(M_1)A = Af(M_2) = Af(-M_2)^{-1}$. Conversely, let T be a unipotent matrix. As we will prove in the next step, there exists $N \in \mathcal{N}$ such that $T = f(N)$. Define Y so that $M_1 A = AM_2 + Y$. Then $\Phi_{M_1, M_2}(Y) = 0$. By Lemma 4.16, $f_1 = 1$ implies $Y = 0$.

Step 4. We must now prove that f actually induces a bijection from $\mathcal{N}_n$ onto $\mathfrak{T}_n$. It is clearly enough to determine a power series h centered in the identity such that $h(f(N)) = N$. In such a case h would be defined on all of $\mathfrak{T}_n$, since $T - I$ is nilpotent whenever $T \in \mathfrak{T}_n$. This implies immediately that f induces an injective map from $\mathfrak{T}_n$ into $\mathcal{N}_n$. Since $|\mathcal{N}_n| = q^{\binom{n}{2}} = |\mathfrak{T}_n|$, clearly f induces also a bijection. Set now $h(y) = \sum_{i \geq 0} h_i (y-1)^i$. Since $f(\mathbf{0}) = I$, then $h_0 = 0$. Substituting formally y with $\sum_{j \geq 0} f_j x^j$, we get

$$\sum_{i \geq 1} h_i \left(\sum_{j \geq 1} f_j x^j \right)^i = x. \tag{26}$$

Collecting x^i in each summand we see that this system is equivalent to a set of equations involving just a finite number of coefficients h_i each time. More precisely, $h_1 = f_1^{-1} = 1$. Again by induction, suppose we have determined every coefficient h_i with $i < k$, then $h_k = s(h_1, \ldots, h_{k-1}, f_j)$, where $0 \leq j < n$, for some term s depending on the coefficients enclosed by the parentheses. ∎

Observe that $M_1 A = AM_2$ is a linear condition in the entries of M_i, hence the number of solutions of this equation over $\mathbb{F}_q$ is actually a q-power and clearly this holds for the corresponding image via f. Specializing M_1, M_2, A, n, we are in the position to prove that the orbit lengths are only q-powers. We start with the linear case, where a slightly more general result holds.

THEOREM 4.19. *Let $\mathfrak{P}$ be the p-Sylow subgroup of $GL(n, q)$, then $\forall g \in \mathfrak{P}$, $|g^{\mathfrak{P}}|$ is a q-power.*

Proof. It is enough to show that $|C_{\mathfrak{P}}(g)|$ is a q-power. Suppose $g = I + N$, where N is a strictly lower triangular matrix, then

$$(I + N)(I + M) = (I + M)(I + N) \Leftrightarrow I + M \in C_{\mathfrak{P}}(g). \tag{27}$$

But this is equivalent to $NM = MN$, which is a linear condition in the entries of M. ∎

Observe that no restriction on p is needed for the preceding result.

We can now prove that the $\mathfrak{T}$-orbit in $\mathfrak{U}$ have only q-power order, completing the proof of Theorem 3.4.

THEOREM 4.20. *$\mathscr{O}(\mathfrak{U}) \subseteq \{q^j \mid 0 \leq j \leq g(n)\}$ unless $\mathfrak{B}$ is symplectic and $p = 2$.*

Proof. It is enough to declare which matrices play the role of M_1, M_2, A and observe that $f(N^t) = f(N)^t$.

(a) In the symplectic case let $M_1 = N^t$, $M_2 = -N$, A symmetric, where all matrices have dimension n.

(b) In the linear case, just apply Theorem 4.19. ∎

Remark 4.1. To show the elegance of this approach, we will prove once again that the matrices described in the symplectic case have the required orbit lengths. Consider first $A = A(1, r) = \left(\begin{smallmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{smallmatrix}\right)$. Then $N^tA + AN = \mathbf{0}$ is equivalent to

$$\begin{pmatrix} N^t & M^t \\ \mathbf{0} & L^t \end{pmatrix}\begin{pmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} + \begin{pmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\begin{pmatrix} N & \mathbf{0} \\ M & L \end{pmatrix} = \mathbf{0}; \tag{28}$$

that is, $N = \mathbf{0}$. Hence this equation has $q^{r(n-r)+\binom{n-r}{2}}$ solutions. But this is also the order of the centralizer of A. It follows that the orbit length is $q^{\binom{n}{2}-\binom{n-r}{2}-r(n-r)} = q^{\binom{r}{2}}$. Let now $A = A(l, r)$. Without restriction we may think that $r = n$. Set $s = n - l - 1$, then

$$\begin{pmatrix} N^t & \mathbf{a} & M^t \\ \mathbf{0} & 0 & \mathbf{b} \\ \mathbf{0} & \mathbf{0} & L^t \end{pmatrix}\begin{pmatrix} I_l & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_s \end{pmatrix} + \begin{pmatrix} I_l & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_s \end{pmatrix}\begin{pmatrix} N & \mathbf{0} & \mathbf{0} \\ \mathbf{a} & 0 & \mathbf{0} \\ M & \mathbf{b}^t & L \end{pmatrix} \tag{29}$$

is the zero matrix and this implies N, M, L, $\mathbf{b}$ must annihilate. The only term surviving is the vector $\mathbf{a}$ which can be chosen in q^l ways, so the corresponding orbit length for the element A is $q^{\binom{l}{2}-l}$.

An interesting question is whether every conjugacy class in these groups has q-power order.

5. CHARACTER DEGREES

We will finally prove that the restriction $q = p$ forces $cd\mathfrak{B} = \mathscr{O}(\mathfrak{U})$.

We provide a correspondence between $\mathfrak{U}$ and $\hat{\mathfrak{U}}$, considered as sets of matrices, which induces a weak $\mathfrak{T}$-equivalence; that is, $\mathscr{O}(\mathfrak{U}) = \mathscr{O}(\hat{\mathfrak{U}})$.

DEFINITION 5.21. Consider

$$A = \sum_{\substack{1 \le i \le k \\ 1 \le j \le r}} a_{ij} E_{ij} \in (K)_{k \times r}$$

and define

$$A^s := \sum_{\substack{1 \le i \le r \\ 1 \le j \le k}} a_{k-j+1,r-i+1} E_{ij} \in (K)_{r \times k}.$$

We will call s the reflection or specular operator (Spiegelungoperator).

Observe that s corresponds to the reflection along the diagonal starting from the bottom left corner. We provide a more manageable description of the reflection operator.

LEMMA 5.22. *Let $J_l = \sum_{i=1}^{l} E_{i,l+1-i}$ and A as above, then $A^s = J_r A^t J_k$.*

Proof. Set $A = \sum_{l,m} a_{lm} E_{lm}$, then

$$\sum_{i=1}^{r} E_{i,r+1-i} \sum_{\substack{1 \le m \le r \\ 1 \le l \le k}} a_{lm} E_{ml} \sum_{j=1}^{k} E_{k+1-jj} = \sum_{\substack{1 \le i \le r \\ 1 \le j \le k}} a_{k+1-j,r+1-i} E_{ij}, \quad (30)$$

as asserted. ∎

Observe that $J_l = J_l^t = J_l^{-1} = J_l^s$.

COROLLARY 5.23. *Suppose dimension matches. Then $(AB)^s = B^s A^s$.*

Proof. Let $A \in (K)_{k \times r}$, $B \in (K)_{r \times m}$. Then $(AB)^s = J_m (AB)^t J_k = J_m B^t J_r J_r A^t J_k = B^s A^s$. ∎

Hence, for $k = r = m$, s is an involutory ring antiautomorphism of $(K)_k$.

COROLLARY 5.24. *Denoting with s and t the reflection and transposition operator, we have $st = ts$.*

Proof. $A^{st} = (J_r A^t J_k)^t = J_k A J_r = A^{ts}$. ∎

For $r = k$, it turns out that st is an automorphism of the central simple algebra $(K)_r$ and, by a theorem of Noether and Skolem, it would be an inner one, precisely that induced by conjugation by J.

The second ingredient of the proof is a lemma establishing how the dual action can be expressed in term of matrices.

DEFINITION 5.25. Given a matrix $A \in (\mathbb{F}_q)_r$, let $\tau(A) = Tr_{\mathbb{F}_q:\mathbb{F}_p}(Tr(A))$.

LEMMA 5.26. *Let $V = (\mathbb{F}_q)_{k \times r}$ and define $\hat{}: V \to Hom(V, \mathbb{F}_p)$ by $\hat{A}(B) = \tau(A^t B)$. Then $\hat{}$ is an $\mathbb{F}_p$-isomorphism.*

Proof. $\mathbb{F}_p$-linearity is clear. By finite dimensionality, it is enough to show that $\hat{}$ is injective. In fact, let $A = \sum_{m,l} a_{ml} E_{lm}$ and $B = bE_{ij}$ then

$$0 = \tau\Big(\sum_{m,l} a_{ml} E_{ml} bE_{ij}\Big) = Tr_{\mathbb{F}_q:\mathbb{F}_p}(ba_{ji}). \tag{31}$$

Since $Tr(\mathbb{F}_q) = \mathbb{F}_p$ and b is arbitrary, this forces $a_{ji} = 0$, $j = 1, \ldots, k$, $i = 1, \ldots, r$. ∎

LEMMA 5.27. *Let $\mathfrak{T} = \{(\begin{smallmatrix} T_1 & 0 \\ 0 & T_2^{-1} \end{smallmatrix})\}$, where T_1, T_2 are unipotent lower triangular matrices of dimension r, k. Let $V = (\mathbb{F}_q)_{k\times r}$ be endowed with a $\mathfrak{T}$-set structure via $A^T = T_2 A T_1$, where $T = (\begin{smallmatrix} T_1 & 0 \\ 0 & T_2^{-1} \end{smallmatrix}) \in \mathfrak{T}$. Then the induced $\mathfrak{T}$-action on $Hom(V, \mathbb{F}_p)$ is described by $\hat{A}^T = \widehat{T_2^* A T_1^*}$. In particular, $C_{\mathfrak{T}}(\hat{A}) = \{T \in \mathfrak{T} : A = T_2^* A T_1^*\}$.*

Proof. We remark that for an arbitrary G-module M, the induced action on the dual space $\hat{M}$ is given by $\mu^g(m) = \mu(m^{g^{-1}})$. Now

$$\begin{aligned}(\hat{A}^T)(B) &= \hat{A}(B^{T^{-1}}) = \tau(A' T_2^{-1} B T_1^{-1}) \\ &= \tau(T_1^{-1} A' T_2^{-1} B) = \tau((T_2^* A T_1^*)' B) \\ &= \widehat{T_2^* A T_1^*}(B).\end{aligned} \tag{32}$$

It follows that $\hat{A}^T = \hat{A}$ if and only if $\hat{A} = \widehat{T_2^* A T_1^*}$. The second part of the lemma is a simple consequence of the injectivity of $\hat{}$. ∎

THEOREM 5.28 (Weak Equivalence). *Let $\mathfrak{T}$ be as in Lemma 5.27 and $\mathfrak{U} = \{(\begin{smallmatrix} I_r & 0 \\ A & I_k \end{smallmatrix}) : A \in (\mathbb{F}_q)_{k\times r}\}$. Then $\mathfrak{U}$ and $\hat{\mathfrak{U}}$ are weakly equivalent $\mathfrak{T}$-sets.*

Proof. $B^{\mathfrak{T}} = \{T_2 B T_1\}$ where T_i varies in the set of unitriangular lower matrices of suitable dimension. Now the $\mathfrak{T}$-orbit of $\widehat{B^{st}} \in \hat{V}$ is

$$\Big\{\widehat{T_2^* B^{st} T_1^*}\Big\} = \Big\{\widehat{\big(T_2^{-s} B T_1^{-s}\big)^{st}}\Big\}.$$

Since the map $T \to T^{-s}$ is an automorphism of the group of unitriangular lower matrices and $\hat{}$ is one-to-one, this set has the same order as the $\mathfrak{T}$-orbit of $B \in V$. Hence $\mathscr{O}(\mathfrak{U}) \subseteq \mathscr{O}(\hat{\mathfrak{U}})$. The converse inclusion follows analogously. ∎

We now consider the symplectic case.

LEMMA 5.29. *Let $\mathfrak{T} = \{(\begin{smallmatrix} T & 0 \\ 0 & T^* \end{smallmatrix})\}$, where T is a unipotent lower triangular matrix of dimension n. Let V, the set of symmetric matrices in $(\mathbb{F}_q)_n$, q odd,* *be endowed with the structure of a* $\mathfrak{T}$*-set via* $A^S = T'AT$, *where* $S = \begin{pmatrix} T & 0 \\ 0 & T^* \end{pmatrix} \in \mathfrak{T}$. *Define* $\hat{}: V \to Hom(V, \mathbb{F}_p)$ *via* $\hat{A}(B) = \tau(A'B)$, *then:*

(a) *For any* $\lambda \in Hom(V, \mathbb{F}_p)$, *and* $B \in V$, $\exists! A \in V$ *such that* $\lambda(B) = \hat{A}(B)$,

(b) $\hat{V} = Hom(V, \mathbb{F}_p)$,

(c) $\hat{A}^T = \widehat{T^{-1}AT^*}$,

(d) $C_{\mathfrak{T}}(\hat{A}) = \{S \in \mathfrak{T} : A = T^{-1}AT^*\}$.

Proof. The part relative to the structure of $Hom(V, \mathbb{F}_p)$ as a $\mathfrak{T}$-set is a particular case of Lemma 5.27 when $k = r = n$ and $T_2 = T_1'$. We need to assure the regularity of the bilinear form $(\,,\,): V \times V \to \mathbb{F}_p$, defined by $(A, B) = \tau(A'B)$. It is enough to show that $\hat{V} \to Hom(V, \mathbb{F}_p)$ is surjective. By Lemma 5.26, any $\mu \in Hom((\mathbb{F}_q)_n, \mathbb{F}_p)$ is of the form $\hat{C}$, where $C \in (\mathbb{F}_q)_n$. We prove that $\lambda = \mu_V$, the restriction of μ to symmetric matrices, is of the form $\hat{A}$ for some symmetric matrix A. Let B be symmetric, then

$$\begin{aligned}(C, B) &= \tau(C'B) = \tau(B'C) \\ &= \tau(CB') = \tau(CB) \\ &= (C', B). \end{aligned} \tag{33}$$

Since q is odd, $(C, B) = (\frac{1}{2}(C' + C), B) = (A, B)$, where $A = \frac{1}{2}(C' + C)$ is the symmetric matrix for which we were looking. ∎

THEOREM 5.30 (Weak Equivalence). *Let* $\mathfrak{T}$ *be as in Lemma* 5.29 *and* $\mathfrak{U} = \{\begin{pmatrix} I & 0 \\ A & I \end{pmatrix} : A \in (\mathbb{F}_q)_n, A' = A\}$. *Then* $\mathfrak{U}$ *and* $\hat{\mathfrak{U}}$ *are weakly equivalent* $\mathfrak{T}$*-sets.*

Proof. We will show that the reflection operator induces a weak equivalence between $\mathfrak{U}$ and its dual. Observe that the operator group is closed under s. The $\mathfrak{T}$-action is given by $T'BT$, where B is symmetric. By Corollary 5.23, we have

$$(T'BT)^s = T^sB^s(T')^s. \tag{34}$$

Set $S^{-1} = T^s$, then, by Corollary 5.24, $S^* = (T')^s$. Hence the reflected image is $S^{-1}B^sS^*$. But, by Lemma 5.29, this term describes the dual action. Since s is a bijection onto $\mathfrak{T}$, this implies that

$$|B^{\mathfrak{T}}| = |(B^s)^{\hat{\mathfrak{T}}}|, \tag{35}$$

where $\hat{}$ suggests that $\mathfrak{T}$ is acting on the dual space. Hence $\mathscr{O}(\mathfrak{U}) \subseteq \mathscr{O}(\hat{\mathfrak{U}})$. By Lemma 5.29, we obtain analogously the converse inclusion. ∎

With the usual restriction $p \neq 2$ when $\mathfrak{P}$ is symplectic, the following statement holds:

COROLLARY 5.31. *Let* $\lambda \in Irr(\mathfrak{U})$, *then* $|I_{\mathfrak{P}}(\lambda):\mathfrak{U}|$ *is a q-power.*

Proof. This follows by weak equivalence and Theorem 4.20. ∎

The necessity of the condition p ≠ 2 in the symplectic case was pointed out to me by Isaacs. One can show that for the $\mathfrak{T}$-set V of symmetric matrices in $(\mathbb{F}_q)_2$, q even, $\mathscr{O}(V) = \{1, q\}$, but $\mathscr{O}(\hat{V}) = cd\mathfrak{P} = \{1, q/2, q\}$. Hence $\mathfrak{U}$, $\hat{\mathfrak{U}}$ are not weakly $\mathfrak{T}$-equivalent and the character degrees conjecture is false in even characteristic. We now show that the restriction q = p implies:

THEOREM 5.32 (Character Degrees). *Suppose* $\mathfrak{P} \in Syl_p(\mathscr{L}_n(\mathbb{F}_p))$, *where* $\mathscr{L}_n(\mathbb{F}_p)$ *is* $SL(n, p)$ *or* $Sp(n, p)$. *Set*

$$g(n) = \begin{cases} f(n), & \mathscr{L}_n(\mathbb{F}_p) \text{ linear}, \\ \binom{\frac{n}{2}}{2}, & \mathscr{L}_n(\mathbb{F}_p) \text{ symplectic}, \end{cases}$$

then $cd\mathfrak{P} = \{p^j : 0 \leq j \leq g(n)\}$.

Proof. Unless the symplectic case in even characteristic, $\mathscr{O}(\mathfrak{U}) = \mathscr{O}(\hat{\mathfrak{U}})$ by Theorems 5.28 and 5.30. By Lemma 2.3, $\mathscr{O}(\hat{\mathfrak{U}}) \subseteq cd\mathfrak{P}$. By Itô's theorem equality holds. In the remaining case, Lemma 5.29 assures only that $\mathscr{O}(\mathfrak{U}) \subseteq \mathscr{O}(\hat{\mathfrak{U}})$, but the restriction $q = 2$ forces again equality. ∎

Isaacs has recently established that the character degrees of the p-Sylow subgroup of $SL(n, q)$ are only q-powers. The author, using a similar procedure, extended this result to the symplectic case in odd characteristic. This shows that the restriction $q = p$ is not necessary in Theorem 5.32.

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