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On a conjecture concerning character degrees of some p-groups

By

ANDREA PREVITALI

1. Introduction. It has been conjectured that the character degrees of $P \in \mathrm{Syl}_p(A_n(q))$, q a p-power, are q-powers. This has recently been established by Isaacs [1]. We extend this result to the p-Sylow subgroups of $C_n(q)$, $D_n(q)$ and ${}^2A_{2n-1}(q^2)$, p an odd prime. This result was proved by Kazhdan [2] under the restriction $n < p$, which allows him to define the exponential map on the Lie algebra $L(P)$ associated to P. On the other hand, Isaacs shows that the character degrees of P are q-powers, whenever P is an algebra group over $\mathbb{F}_q$. Unfortunately, it is not clear whether this applies to our cases. To solve this problem, we define a map sharing some of the properties of the exponential map, with the advantage of being defined whenever p is odd, and prove that some suitable sections of P are strong subgroups.

2. Preliminary results. Let G denote one of the above mentioned groups of Lie type and set $F = \mathbb{F}_{q^m}$, where $m = 2$ when $G = {}^2A_{2n-1}(q^2)$ and $m = 1$ otherwise. Let α be the identity automorphism of $\mathbb{F}_q$ for $G = C_n(q)$, $D_n(q)$ or the conjugation $x^\alpha = \bar{x} = x^q$ in $\mathbb{F}_{q^2}$, when $G = {}^2A_{2n-1}(q^2)$. Define β on $(F)_n$ by $A^\beta = A$, $-A$ or $-\bar{A}$ for $G = C_n(q)$, $D_n(q)$ or ${}^2A_{2n-1}(q^2)$. Let T^* denote $(T^t)^{-1}$ and $U_n(F)$ the p-Sylow subgroup of $SL(n, F)$ constituted by unitriangular lower matrices. E_{ij} stands for the elementary matrix with entry 1 in the position (i, j), zero otherwise. We recall some of Isaacs' definitions. Given a group P, we say that P is an *algebra group* over F, if $P = 1 + J(A)$, where A is an associative finite dimensional algebra over the field F and $J(A)$ denotes its Jacobson radical. We say that C is an *algebra subgroup* of P if $C \leqq P$ and C is an algebra group. S is a *strong* subgroup of P if $|S \cap C|$ is a q-power for every algebra subgroup C of P. We say that P is a *q-power-degree group* if the degree of every irreducible character of P is a q-power.

Lemma 1. *Let G be one of $C_n(q)$, $D_n(q)$ or ${}^2A_{2n-1}(q^2)$ and define the groups* $N = \left\{\begin{pmatrix} I & 0 \\ A & I \end{pmatrix} : A^t = A^\beta \in (F)_n\right\}$, $K = \left\{\begin{pmatrix} T^\alpha & 0 \\ 0 & T^* \end{pmatrix} \quad T \in U_n(F)\right\}$. *Then K normalizes N, $N \cap K = 1$ and $P = NK$ is a Sylow p-subgroup of G.*

P r o o f. It is enough to count the order of KN and observe that the matrices of this form preserve the scalar product defined by the matrix $E = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ in the symplectic case and by $E = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$ in the orthogonal and unitary ones. □

We now determine the structure of the inertia subgroup $I_P(\lambda)$, where $\lambda \in \mathrm{Irr}(N)$. Denote with Tr the trace map from F onto $\mathbb{F}_p$ and set $\tau(A) = \mathrm{Tr}(\sum a_{ii})$, where $A = (a_{ij}) \in (F)_n$.

Lemma 2. *Let $V = \{A \in (F)_n : A^\beta = A^t\}$, where F is a finite field of odd characteristic. Let V be endowed with a K-set structure via $A^R = T^t A T^\alpha$, where $R = \begin{pmatrix} T^\alpha & 0 \\ 0 & T^* \end{pmatrix}$*

Define $\phi: (F)_n \longrightarrow \mathrm{Hom}((F)_n, \mathbb{F}_p)$ via $A^\phi(B) = \tau(AB)$, then

1. $V^\phi = \mathrm{Hom}(V, \mathbb{F}_p)$,
2. $(A^\phi)^R = (T^{-\alpha} A T^*)^\phi$,
3. $C_K(A^\phi) = \{R \in K : A = T^\alpha A T^t\}$.

Proof. Set $W = (F)_n$. We first prove that ϕ is a $\mathbb{F}_p$-isomorphism between W and $\mathrm{Hom}(W, \mathbb{F}_p)$. $\mathbb{F}_p$-linearity is clear. By finite dimensionality, it suffices to show that ϕ is injective. In fact consider $A = \sum_{m,l} a_{ml} E_{ml}$ and $B = b E_{ij}$ in W, then

$$0 = \tau\left(\sum_{m,l} a_{ml} E_{ml} b E_{ij}\right) = \mathrm{Tr}(b a_{ji}). \tag{1}$$

Since $\mathrm{Tr}(F) = \mathbb{F}_p$ and b is arbitrary, this forces $a_{ji} = 0$, hence $A = 0$. We prove $V^\phi = \mathrm{Hom}(V, \mathbb{F}_p)$ when V is the set of symmetric matrices in W. Since $W^\phi = \mathrm{Hom}(W, \mathbb{F}_p)$, any $\mu \in \mathrm{Hom}(W, \mathbb{F}_p)$ is of the form C^ϕ with $C \in W$. We show that $\lambda = \mu_V$, the restriction of μ to V, is of the form A^ϕ for $A \in V$. Let B be an element in V, then

$$\tau(CB) = \tau(B^t C^t) = \tau(BC^t) = \tau(C^t B).$$

Since q is odd, $\tau(CB) = \tau\left(\frac{1}{2}(C^t + C)B\right) = \tau(AB)$, where $A = \frac{1}{2}(C^t + C)$ is the symmetric matrix we were looking for. The orthogonal case follows in the same way and, for the unitary one, we need only to observe that $\mathbb{F}_p$ is elementwise fixed by conjugation.

We finally remark that for an arbitrary G-module M, the induced action on the dual space $\hat{M}$ is given by $\mu^g(m) = \mu(m^{g^{-1}})$. Now

$$\begin{aligned}(A^\phi)^R(B) &= (A^\phi)(B^{R^{-1}}) = \tau(AT^* B T^{-\alpha}) \\ &= \tau(T^{-\alpha} A T^* B) = (T^{-\alpha} A T^*)^\phi(B),\end{aligned}$$

proving the last two statements. □

Let ε denote a primitive p-th root of unity in $\mathbb{C}$. Apart from composition with the map defined by $v(a) = \varepsilon^a$, for $a \in \mathbb{F}_p$, we can identify A^ϕ with a linear character of N. With this abuse of notation the following statement holds

Corollary 1. $I_P(A^\phi)/N \cong \{T \in U_n(F) : T^\alpha A T^t = A\}$.

Proof. $\{T \in U_n(F) : T^\alpha A T^t = A\} \cong C_K(A^\phi) \cong I_P(A^\phi)/N$.

By classical Clifford's techniques, since

$$1 \longrightarrow N \longrightarrow I_P(\lambda) \longrightarrow I_P(\lambda)/N \longrightarrow 1$$

is a splitting sequence for all $\lambda \in \operatorname{Irr}(N)$, λ is extendible to a character $\lambda_0 \in I_P(\lambda)$ and, by Gallagher's theorem,

$$\operatorname{Irr}(I_P(\lambda) \mid \lambda) = \{\lambda_0 \mu : \mu \in \operatorname{Irr}(I_P(\lambda)/N)\}.$$

Hence P is a q-power-degree group if this holds for $I_P(\lambda)/N$ and if $I_P(\lambda)$ has q-power index in P for every $\lambda \in \operatorname{Irr}(N)$. Following Isaacs' paper, this follows once we show that $I_P(\lambda)/N$ is a strong subgroup. For this purpose, we define a map f from $\mathcal{N}_n$, the nilpotent algebra of lower triangular matrices of dimension n over F onto $U_n(F) = I + \mathcal{N}_n$.

Lemma 3. *Set* $C_n = \frac{1}{n+1}\binom{2n}{n}$, *then* C_n *is an integer* (C_n *is also called the n-th Catalan number*).

Proof. C_n satisfy the quadratic recurrence relation $C_n = \sum_{k=1}^{n} C_{k-1} C_{n-k}$ and $C_0 = 1$ (see [3], page 41), hence C_n is an integer. □

Theorem 1. *Let F be a finite field of odd characteristic, $\mathcal{N}_n$ the nilpotent algebra of strictly lower triangular matrices over F and $U_n(F) = I + \mathcal{N}_n$, then there exists $f \in F[[x]]$ satisfying the following conditions*:

1. $\forall X \in \mathcal{N}_n, f(-X) = f(X)^{-1}$.
2. *f induces a bijection from $\mathcal{N}_n$ onto $U_n(F)$.*
3. $\forall A \in (F)_n$, $X, Y \in \mathcal{N}_n$, $f(Y) A f(X) = A \Leftrightarrow YA + AX = 0$.

Proof. Set $f(x) = \sqrt{ + x^2} + x$, where the determination of the sign is such that $(0) = 1$. Formally

$$f(x) = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} x^{2k} + x.$$

Now

$$\binom{\frac{1}{2}}{k} = \frac{(-1)^{k-1}}{2^{2k-1}} \frac{1}{k} \binom{2k-2}{k-1} = \frac{(-1)^{k-1}}{2^{2k-1}} C_{k-1},$$

where $C_{-1} = -\frac{1}{2}$.

By Lemma 3

$$(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{2^{2k-1}} C_{k-1} x^{2k} + x$$

is well-defined, whenever the characteristic is odd. For any $X \in \mathcal{N}_n$, $f(X)$ exists and belongs to $C(X)$, the centralizer of X in $U_n(F)$, and

$$f(X) f(-X) = (\sqrt{I + X^2} + X)(\sqrt{I + X^2} - X) = I.$$

Set $f(X) = T$, then $I + X^2 = T^2 - 2TX + X^2$ and $X = \frac{T - T^{-1}}{2}$, hence f is injective. Since $\mathcal{N}_n$ is finite f induces a bijection.

Suppose $SAT = A$, where $T = f(X)$ and $S = F(Y)$, then $X = \dfrac{T - T^{-1}}{2}$, $Y = \dfrac{S - S^{-1}}{2}$ and

$$YA + AX = \frac{1}{2}(SA - AS^{-1} + AT - AT^{-1}) = 0.$$

Conversely, if $YA = -AX$, then, for every power series g, we have

$$g(Y)A = Ag(-X).$$

In particular, for $g = f$, we get $f(Y)Af(X) = A$. □

Lemma 4. *Suppose $C = I + L$ is an algebra subgroup of $U_n(F)$, where L is a subalgebra of $\mathcal{N}_n$, then $C = f(L)$.*

Proof. Since L is an algebra, $f(L) \subseteq I + L$. The converse inclusion follows from $|f(L)| = |L| = |I + L|$. □

Corollary 2. *$S = I_P(\lambda)/N$ is a strong subgroup for every $\lambda \in \mathrm{Irr}(N)$.*

Proof. By Corollary 1 and Theorem 1, we know that, up to isomorphism, $S = f(M)$, where M is the $\mathbb{F}_q$-vector space $\{Z \in \mathcal{N}_n \colon Z^\alpha A + AZ^t = 0\}$ and $\lambda = A^\phi$. Let $C = I + L$ be an algebra subgroup of $U_n(F)$, then $S \cap C = f(M \cap L)$. But $M \cap L$ is a $\mathbb{F}_q$-vector space, so its order is a q-power. Since f is one-to-one, this holds for $|S \cap C|$ also. □

By Theorem D in [1], it follows that S is a q-power-degree group and S has q-power order. By the remarks preceding Theorem 1, we obtain

Theorem 2. *If $P \in \mathrm{Syl}_p(G)$, $G = C_n(q)$, $D_n(q)$, or ${}^2A_{2n-1}(q^2)$, q odd, then P is a q-power-degree group.*

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Anschrift des Autors:

Andrea Previtali
Via S. Eurosia, 15
I-22064 Casatenovo (CO)
Italia

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