

Another Existence and Uniqueness Proof for the Higman–Sims Simple Group*

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Abstract. In this article, we give a short proof for the existence and uniqueness of the Higman–Sims sporadic simple group HS by means of the first author's algorithm [17] and uniqueness criterion [18], respectively. We realize HS as a subgroup of $\text{GL}_{22}(11)$, and determine its automorphism group $\text{Aut}(\text{HS})$. We also give a presentation for $\text{Aut}(\text{HS})$ in terms of generators and relations. Furthermore, the character table of HS is determined and representatives of its conjugacy classes are given as short words in its generating matrices inside $\text{GL}_{22}(11)$.

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1 Introduction

Originally, Higman and Sims [11] constructed their sporadic simple group HS as a rank 3 primitive permutation group of degree 100. HS was shown to be the unique simple group of its order by Parrott and Wong [19] and classified by the structure of the centralizer $H = C_G(z) \cong (4 * 2^{1+4})S_5$ of a 2-central involution z of $G = \text{HS}$ by Janko and Wong in [14]. It is worth mentioning that Gorenstein and Harris [9] were able to show that HS is the unique simple group whose Sylow 2-subgroup is isomorphic to a split extension of $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$ by a dihedral group of order 8. The arguments given in these important articles are involved and do not generalize to other sporadic groups.

Therefore, a new existence and uniqueness proof for the Higman–Sims group is given here using the first author's algorithm [17] and uniqueness criterion [18] which together provide foundations for uniform existence and uniqueness proofs for all

sporadic simple groups characterizable by a centralizer H of a 2-central involution. In each application one starts from a given presentation of H in terms of generators and relations. Thus, ambiguities are avoided (see introduction of Section 2). Since these groups H are fairly small, computer algebra systems like MAGMA [2] and GAP [20] can be used to find the conjugacy classes, character tables and suitable subgroups of H . So, long and difficult arguments of the original papers can be replaced by well-documented statements, which are easy to check. Furthermore, in our uniqueness proof for the Higman–Sims group HS, we develop for the first time an abstract classification of all the conjugacy classes in any finite simple group G of HS-type defined by:

Definition 1.1. A finite simple group X is called to be of *HS-type* if it possesses a 2-central involution z such that $C_X(z) \cong H$, where H is the group of even order defined by generators and relations in Lemma 2.1.

In Section 3, we first study the fusion of the involutions of H in any finite group G of HS-type. Then we determine the group order of G and the structure of the second centralizer $C_G(v)$. With this information the normalizers of the cyclic subgroups of order $p \in \{2, 3, 5, 7\}$ of a group G are constructed. Furthermore, the fusion of their conjugacy classes in G is shown to be uniquely determined (see Proposition 3.11).

In Section 5, we use these results to show that each finite simple group G of HS-type has a unique irreducible character χ of degree $\chi(1) = 22$ (see Proposition 5.1). Using then the first author’s uniqueness criterion [18], we prove in Theorem 5.2 that each finite simple group G of HS-type is isomorphic to the simple group $\mathfrak{G} = \langle \mathcal{R}, \mathcal{S}, \mathcal{C}, \mathcal{G} \rangle \leq \text{GL}_{22}(11)$ of order $|\mathfrak{G}| = 2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$ constructed in Theorem 4.2. The generating matrices of \mathfrak{G} are stated there as well. In Section 6, we determine the automorphism group $\text{Aut}(\mathfrak{G}) = \langle \mathfrak{G}, \Gamma \rangle \leq \text{GL}_{22}(11)$ of the simple group \mathfrak{G} of HS-type (see Theorem 6.1). Finally, in Theorem 6.2, a presentation of $\text{Aut}(\mathfrak{G})$ is given in terms of generators and relations. This will be applied in [16].

Most of the calculations were done with MAGMA [2] and GAP [20]. In order to document our results, we provide the list of representatives of the conjugacy classes and the character tables of the local subgroups $H = C_G(z)$, $U = C_G(v)$, $N_G(3_A)$, etc. in Appendices A and B, respectively. Hence, $N_G(3_A)$ or $N_G(e)$ denotes the normalizer of the cyclic group $\langle e \rangle$, where e is a representative of the conjugacy class 3_A of G . Concerning notation and terminology we refer to the books by B. Huppert [12], I.M. Isaacs [13], and M. Suzuki [21].

2 Structure of the Given Centralizer

As mentioned by Janko and Wong [14], the given centralizer H is an extension of a non-abelian group $T \cong 4 * 2^{1+4}$ of order 64 by the symmetric group S_5 . J. Cannon informed the second author that the command `ExtensionsOfSolubleGroup` of MAGMA [2] produces forty eight non-isomorphic extension groups H_i with $O_2(H_i) \cong T$, and $H_i/O_2(H_i) \cong S_5$, where $O_2(X)$ denotes the largest normal subgroup of 2-power order of the group X . In order to avoid ambiguities, we give a presentation of that extension that will be shown to lead to the Higman–Sims

group.

Lemma 2.1. *Let H be the finite group of even order generated by elements r, s, c subject to the following set $\mathcal{R}(H)$ of relations:*

$$\begin{aligned} r^3 &= s^8 = c^5 = 1, \\ (r^{-1}s^{-1})^4 &= (rc^{-1}r^{-1}s^{-1})^2 = (r^{-1}cs^{-2})^2 = (r^{-1}cs^2)^2 = 1, \\ rcr &= sc^{-2}rs, \quad scrcsr = c^2, \quad s^2cr = rcsrcs. \end{aligned}$$

Let $f = srcsr^{-1}sc$, $e = rscsc^2$, $t_2 = r^{-1}c^2scs$, $z_2 = (rc^{-1})^3$, $u = u_2^2$, $t = t_2^2$, $z = z_2^2$, $b_1 = s^{-1}rs^{-1}r^{-1}s^{-1}rs$, $b_2 = (sr)^2$, and $k = rsr^{-1}sr^{-1}s^{-1}$. Then f, u, t, z, b_1, b_2, k are involutions, u_2, t_2, z_2 have order 4, and the following assertions hold:

- (a) H has an outer automorphism γ defined by $r^\gamma = r$, $s^\gamma = rsr^2s^2rsr^2s^{-1}$, $c^\gamma = rc^2r^{-1}c^2r$.
- (b) $S = \langle s, f, s^2, u_2, t_2, z_2, u, t, z \rangle$ is a Sylow 2-subgroup of H with center $Z(S) = \langle z \rangle$ such that $\gamma(S) = S$. Moreover, $s^4 = z_2^2 = z$, $u_2^2 = u$, $t_2^2 = t$, $z_2^2 = z$, and $u^2 = t^2 = z^2 = 1$.
- (c) $P = \text{O}_2(H) = \langle z_2, t, u, b_1, b_2 \rangle = \langle z_2 \rangle * \langle t, u, b_1, b_2 \rangle \cong 4 * 2^{1+4}$ with center $Z(P) = \langle z_2 \rangle$.
- (d) $A = \langle z, t, u \rangle$ is the only elementary abelian normal subgroup of order 2^3 of S , and $\gamma(u) = u$, $\gamma(t) = t$, $\gamma(z) = z$.
- (e) $V = \langle z, t \rangle$ is the unique normal Klein four subgroup of S .
- (f) $C_H(V) = C_S(V) = \langle f, s^2, u_2, t_2, z_2, u, t, z \rangle$ and $Z(C_S(V)) = V$.
- (g) $N_H(V) = S$.
- (h) $C = C_H(A) = C_S(A) = \langle u_2, t_2, z_2 \rangle$ is a homocyclic group isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$.
- (i) C is the unique homocyclic rank 3 subgroup of order 64 in S .
- (j) Set $D = N_H(A)$, then $A \leq \Phi(D)$, the Frattini subgroup of D . In particular, A is not complemented in D . Moreover, $D = \langle r, s \rangle$.
- (k) $D = C : K$, where $K = \langle r, k \rangle$ is isomorphic to the symmetric group S_4 on four letters. Moreover, $z^r = z$, $t^r = u$, $u^r = tu$, $z^k = z$, $t^k = ztu$, $u^k = u$.
- (l) $\text{Out}(H) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.
- (m) H has a faithful permutation representation of degree 80 with stabilizer $H_1 = \langle sr^{-1}s^{-1}c^2s^{-1}, scs^{-2}cs \rangle$ of order 96.
- (n) A system of representatives h_i and the corresponding centralizer orders $|C_H(h_i)|$ of the twenty eight conjugacy classes h_i^H of H is given in Appendix A.1.
- (o) The character table of H is given in Appendix B.1.
- (p) $\text{Out}(D)$ is elementary abelian of order 8.
- (q) A system of representatives d_i and the corresponding centralizer orders $|C_D(d_i)|$ of the twenty seven conjugacy classes d_i^D of $D = \langle r, s \rangle$ is given in Appendix A.2.
- (r) The character table of D is given in Appendix B.2.

Proof. All assertions have been checked by means of GAP [20] and MAGMA [2]. □

3 Fusion

In this section, we determine the conjugacy classes of a finite simple group of HS-type. The proofs of the following two subsidiary results use some ideas of Janko and Wong [14].

Proposition 3.1. *Let G be a finite simple group of HS-type having a 2-central involution z with centralizer $H = C_G(z) = \langle r, s, c \rangle$ defined in Lemma 2.1. Let $v = (rs^{-1})^4 s^2$. Then the following statements hold:*

- (a) v is an involution of the Sylow 2-group S defined in Lemma 2.1 with centralizer $C_H(v)$ of order 2^6 .
- (b) $T = C_H(v)$ is a self-normalizing subgroup of $C_G(v)$.
- (c) T is a Sylow 2-subgroup of $C_G(v)$.
- (d) z and v are not conjugate in G .

Proof. (a) Using MAGMA and the faithful permutation representation of H of degree 80 described in Lemma 2.1(m), it is easy to see that $v \in S$ and $|C_H(v)| = 2^6$.

(b) Suppose that the subgroup R of $C_G(v)$ normalizes T . By MAGMA,

$$Z(T) \cap T' = \langle z \rangle,$$

so $\langle z \rangle \triangleleft R$. Hence, $R \leq C_G(z) \cap C_G(v) = C_H(v) = T$. Thus, $R = T$.

(c) It follows immediately from (b).

(d) If z and v were conjugate in G , then $|C_G(v)| = |C_G(z)| = 2^9 \cdot 3 \cdot 5$ by Lemma 2.1, which is impossible by assertions (a) and (c). □

Proposition 3.2. *Let G be a finite simple group of HS-type having a 2-central involution z with centralizer $H = C_G(z) = \langle r, s, c \rangle$. Let S be a Sylow 2-subgroup of H , let $A = \langle z, t, u \rangle$ be the unique elementary abelian normal subgroup of S with order $|A| = 8$ and normalizer $D = N_H(A)$, and let $C = C_H(A)$, where S and A are defined in Lemma 2.1. Then the following statements hold:*

- (a) $E = N_G(A) = C : L$ with $L \cong L_3(2)$. In particular, there exists an element g of order 3 in $N_G(A) \setminus N_H(A)$ such that $z^g = t$.
- (b) $E = N_G(A) = C : L$, where $L = \langle r, g \rangle \cong L_3(2)$.
- (c) The automorphism group $\text{Aut}(D)$ consists of one (H^*, E^*) -double coset, where $H^* = N_{\text{Aut}(H)}(D)/C_{\text{Aut}(H)}(D)$ and $E^* = N_{\text{Aut}(E)}(D)/C_{\text{Aut}(E)}(D)$.
- (d) $E = \langle r, s, g \rangle$ with set $\mathcal{R}(E)$ of defining relations: $r^3 = s^8 = g^3 = 1$, $(gs)^2 = (sr)^4 = (gr)^4 = 1$, $[s^4, r] = 1$, $rg^{-1}s^{-2}r^{-1}g^{-1}rg^{-1}s = 1$, and $sg^{-1}s^{-2}r^{-1}s^2rg^{-1} = 1$.
- (e) E has a faithful permutation representation of degree 28 with stabilizer $E_1 \langle x, y \rangle \leq E$, where $x = r^{-1}g^{-1}rg^{-1}r^{-1}g^{-1}s$ has order 3 and $y = sg^{-1}r^{-1}g^{-1}rs^{-1}g$ has order 4.
- (f) A system of representatives (e_i) and the corresponding centralizer orders $|C_E(e_i)|$ of the nineteen conjugacy classes of e_i^E of $E = \langle r, s, g \rangle$ are given in Appendix A.3.

(g) The character table of E is given in Appendix B.3.

Proof. (a) Let $w = (rs)^2s(r^2s)$. Then w is an involution of H , but $w \notin H'$. As G is simple Thompson’s transfer, Lemma 1.8 of [22, p. 127] implies that there is an element $g \in G$ such that w^g belongs to one of the conjugacy classes $2_a, 2_b$ or 2_e of H determined in Lemma 2.1(c). From Appendix A.1, it follows that $C_G(w)$ has a Sylow 2-subgroup S_1 of order $|S_1| \geq 2^6$.

Using the faithful permutation representation of H described in Lemma 2.1(m), it follows that

$$C_H(w) = \langle w, z \rangle \times \langle s_1, s_2 \rangle : \langle d, s_3 \rangle \cong \langle w, z \rangle \times S_4,$$

where $s_1 = rs^2cr^{-1}c^{-1}r$, $s_2 = rs^{-1}r^{-1}s^{-1}r$, $s_3 = r^{-1}c^{-1}rsc$ are involutions, $d = sr s^3 c^{-1}$ has order 3, and the following defining relations of the symmetric group $S_4 = \langle s_1, s_2, s_3, d \rangle$ hold: $s_1^{s_3} = s_2$, $s_1^d = s_1 s_2$, $s_2^d = s_1$ and $d^{s_3} = d$. Furthermore, $s_1 s_2 = t$ is in the center of the dihedral group $\langle s_1, s_2, s_3 \rangle$ of order 8. Hence, $S_2 = \langle w, z \rangle \times \langle s_1, s_2, s_3 \rangle$ is a Sylow 2-subgroup of $C_H(w)$ with order 2^5 and center $Z(S_2) = \langle w, z, t \rangle$ of order 8.

Therefore, S_2 is a proper normal subgroup of a 2-subgroup V of $C_G(w)$ such that $|V : S_2| = 2$ because $|S_1| > |S_2|$. Hence, there is an element $b \in V$ with $b \notin S_2$, and so $b \notin H$. Thus, $z^b \neq z$. As $Z(S_2)$ is a characteristic subgroup of S_2 , it is normal in V_1 . Hence, b normalizes $Z(S_2)$. Observe (e.g., by MAGMA) that the seven involutions of $Z(S_2)$ fall into three different H -orbits $\{z\}$, $B_1 = \{zt, t\}$ and $B_2 = \{w, wz, wt, wzt\}$. If (a) were false, then $z^b \in B_2$ and so b normalizes the set B_1 ; but then b centralizes $(zt)t = z$, a contradiction. Hence, the claim in (a).

(c) By (a), there is an element $x \in G$ such that $z^x = t$. Let $K = C_G(t)$. Then S^x is a Sylow 2-subgroup of $K = H^x$. By Lemma 2.1, $X = C_S(V) = C_H(t)$ has order 2^8 . Now Sylow’s Theorem asserts that there are a Sylow 2-subgroup Y of K containing X and an element $y \in K$ such that $Y = S^{xy}$. As $V = \langle z, t \rangle = Z[C_S(V)] = Z(X)$ by Lemma 2.1(d), the Klein four group V is normal in $Y = S^{xy}$. Hence, $V = V^{xy}$ because V is the unique normal Klein four subgroup of S by Lemma 2.1(c).

Therefore, $xy \in N_G(V)$, and $z^{xy} = t^y = t$ because $y \in K = C_G(t)$. In particular, $xy \notin H$. Furthermore, Lemma 2.1 asserts that

$$C = C_H(A) = C_S(A) \leq C_G(V) \trianglelefteq N_G(V),$$

and C is the unique homocyclic subgroup of order 64 in S with 2-rank 3. Hence, $C, C^{xy} \leq C_G(V) \leq S$, and so $C^{xy} = C$, and $g \in N_G(C)$. Since $A = \Omega_1(C)$ is a characteristic subgroup of C , it follows that $xy \in N_G(A)$.

Let $D = N_H(A)$. Then Lemma 2.1(j) asserts that D splits over C , and $D/C \cong S_4$. Since D contains a Sylow 2-subgroup of G , the index $|N_G(A) : D|$ is odd. Therefore, Gaschütz’s Theorem [12, p. 121] asserts that $E = N_G(A) = C : L$, where L is a complement of the normal subgroup $C = C_G(A)$ of E . As S_4 is a maximal subgroup of

$$L_3(2) \cong \text{Aut}(A) \geq N_G(A)/C_G(A) > D/C_G(A) \cong S_4,$$

it follows that $E = N_G(A) = C : L \cong C : L_3(2)$.

(b) Since $L \cong L_3(2)$ acts transitively on the set of seven hyperplanes of A and as $C_G(A)$ is a 2-group, the claim is obvious.

(d) An application of MAGMA shows that $\text{Aut}(D)$ has only one (H^*, E^*) -double coset. From this it follows that there is a unique embedding of D in E . This may also be checked using `ExtensionsOfSolubleGroup` in MAGMA which shows that there is a unique split extension E of C by $L_3(2)$ such that $C_E(C) = C$. This provides the proof for (e). Moreover, the action of g on C is given by

$$\begin{pmatrix} 0 & 1 & 2 \\ 3 & 3 & 2 \\ 2 & 0 & 1 \end{pmatrix} \in \text{GL}_3(\mathbb{Z}_4).$$

The remaining assertions (e)–(h) are easily checked by MAGMA. □

Definition 3.3. Let $T = C_G(x)$ be the centralizer of an element x of prime p order in the group G . A representative t of order ps of a conjugacy class of T is called *x-special* if $t^s = x$.

Observe that two x -special elements of G are conjugate in G if and only if they are conjugate in $C_G(x)$.

Proposition 3.4. *Let G be any finite simple group of HS-type having a 2-central involution z with centralizer $C_G(z) = H$. Then the following statements hold:*

- (a) G has two conjugacy classes of involutions represented by z and v or 2_A and 2_B , respectively.
- (b) G has twelve z -special conjugacy classes represented by the following conjugacy classes of H :

$$2_a, 4_a, 4_b, 4_e, 6_a, 8_a, 8_d, 8_e, 10_a, 12_a, 20_a, 20_b.$$

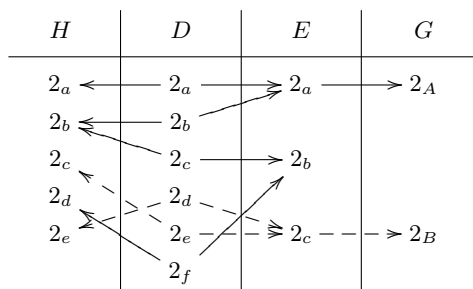
As conjugacy classes of G , they are represented by $2_A, 4_A, 4_B$, etc.

- (c) *Using the notation of the conjugacy classes of $H = C_G(z)$ and $E = N_G(B)$ as in Appendices A.1 and A.3, their fusion patterns into the special conjugacy classes of G are given by the following table:*

G	2_A			2_B		4_A		4_B				4_C		8_A	8_B			8_C
H	2_a	2_b	2_d	2_c	2_e	4_a	4_d	4_b	4_c	4_f	4_h	4_e	4_g	8_a	8_b	8_c	8_d	8_e
E	2_a	2_b		2_c		4_a	4_b	4_d	4_f		4_c	4_e	8_a	8_b			8_c	

Proof. (a) By Proposition 3.1(a), the involutions z and v of G are non-conjugate. Suppose that the classes 2_A and 2_B of involutions of G are represented by z and v , respectively. Let $A = \langle z, t, u \rangle$ be the unique maximal elementary abelian normal subgroup of the fixed Sylow 2-subgroup S of H . Let $D = N_H(A)$ and $E = N_G(A)$. By Proposition 3.2(d), $\text{Aut}(D)$ has a unique (H^*, E^*) -double coset. Therefore, statement (2.7) of Goldschmidt [6, p. 381] asserts that the amalgam $H \leftarrow D \rightarrow E$ is uniquely determined up to isomorphism. In particular, in all groups G of HS-type, there is a unique fusion pattern between the conjugacy classes of 2-elements of D and the ones in H and E , respectively.

Using now the faithful permutation representations of H and E described in Lemma 2.1, Proposition 3.2 and MAGMA, it follows that the six conjugacy classes of involutions of D fuse to the five conjugacy classes of involutions of H , and to the three conjugacy classes of involutions of E , respectively, as shown by the following diagram:



This fusion graph shows that G has exactly two conjugacy classes of involutions.

(c) The fusion of the other conjugacy classes of 2-elements of D in H and E is dealt with by the same methods.

(b) The z -special conjugacy classes of G are then easily obtained from Lemma 2.1(c). □

Proposition 3.5. *Let G be a finite simple group of HS-type having a 2-central involution z with centralizer $H = C_G(z) = \langle r, s, c \rangle$ defined in Lemma 2.1. Let $g \in G$ be as in Proposition 3.2(a), $v = (rs^{-1})^4 s^2$, and $U = C_G(v)$. Then the following statements hold:*

- (a) U does not have a proper normal subgroup of odd index.
- (b) $O(U) = 1$.
- (c) $U = \langle v \rangle \times J$, where $J \cong \text{Aut}(A_6)$.
- (d) Let $y_1 = srs^{-1}r^{-1}srs^2$, $y_2 = y_1s^4$, $y_3 = s^2rs^{-1}r^{-1}srs$, $y_4 = sr^{-1}s^2rs$ and $e = (srs^3c^{-1})^{g^2}$. Then $z = y_1y_2$, $t = y_1y_3^{y_4}$, and $J = \langle y_1, y_2, y_3, y_4, e \rangle$ has the following set $\mathcal{R}(J)$ of defining relations: $y_i^2 = 1$ for $1 \leq i \leq 4$, $[z, y_j] = [t, y_j] = 1$ for $1 \leq j \leq 3$, $t^{y_4} = tz$, $[y_1, y_2] = 1$, $y_1^{y_3} = y_2$, $y_1^{y_4} = tzy_3$, $y_2^{y_4} = ty_3$, $y_3^{y_4} = ty_1$, $y_1^e = y_1y_2$, $y_2^e = y_1$, $[tz, e] = 1$, $e^{y_3} = e^2$, $(e^{y_4}e)^5 = 1$, $[y_4, e]^4 = 1$.
- (e) U has a faithful permutation representation of degree 80 with stabilizer $U_1 = \langle e^{-1}y_4y_1ey_4e, vy_4y_1e^{-1}y_4ey_4e^{-1}y_4 \rangle$.
- (f) A system of representatives u_i and the corresponding centralizer orders $|C_U(u_i)|$ of the twenty six conjugacy classes $(u_i)^U$ of $U = \langle v, y_1, y_3, y_4, e \rangle$ are given in Appendix A.4.
- (g) G has three v -special conjugacy classes represented by the U -classes $2_a, 6_a, 10_c$.
- (h) Using the classification of the conjugacy classes of elements of 2-power order of $H = C_G(z)$ and $U = C_G(v)$ given in Appendices A.1 and A.4, respectively, their fusion patterns into the special conjugacy classes of G are given by the following table:

G	2_A			2_B				4_A	4_B	4_C				8_A	8_B	8_C			
H	2_a	2_b	2_d	2_c	2_e			4_a	4_b	4_e				8_a		8_d	8_e		
U	2_b	2_f		2_a	2_c	2_d	2_e	2_g	4_c	4_d	4_a	4_b	4_e	4_f	8_c	8_d		8_a	8_b

Proof. (a) Let N be the smallest normal subgroup of U of odd index $|U : N|$. By Proposition 3.1(c), $T = C_H(v)$ is a Sylow 2-subgroup of U . Hence, T is a Sylow 2-subgroup of N . Therefore, $U = N_U(T)N = TN = N$ by the Frattini argument and Proposition 3.1(b).

(b) Using the faithful permutation representation of H described in Lemma 2.1(b) and MAGMA, it follows that $\langle v \rangle$ is complemented in T . Since T is a Sylow 2-subgroup of U , Gaschütz’s Theorem [12, p. 121] asserts that $U = \langle v \rangle \times J$. Then $W = J \cap H$ is a Sylow 2-subgroup of J . Furthermore, W is generated by the involutions $y_i, i = 1, \dots, 4$. It also follows that $z = y_1y_2, t = y_1y_3^{y_4}$, and these generators of W satisfy the following set $\mathcal{R}(W)$ of relations: $y_i^2 = 1$ for $1 \leq i \leq 4$, $[z, y_j] = [t, y_j] = 1$ for $1 \leq j \leq 3, t^{y_4} = tz, [y_1, y_2] = 1, y_1^{y_3} = y_2, y_1^{y_4} = ty_3, y_2^{y_4} = ty_3$ and $y_3^{y_4} = ty_1$.

Let $w = (rs)^2s(r^2s)^2$. Using MAGMA again, it can be checked that $w = v^g$, where g is defined in Proposition 3.2, and

$$C_H(w) = \langle w \rangle \times \langle z \rangle \times \langle y_1^g, y_2^g \rangle : \langle e^g, y_3^g \rangle$$

for some element $e \in U$ of order 3 satisfying the following relations: $y_1^e = y_1y_2, y_2^e = y_1$ and $e^{y_3} = e^2$.

Since $y_1y_2 = z$, it follows that all three involutions y_1, y_2, z are conjugate in U .

Now let $R = O(U)$. Then the Klein four group $Y = \langle y_1, y_2 \rangle$ operates on R . Applying the Brauer–Wielandt formula we get

$$|R| \cdot |C_R(Y)|^2 = |C_R(y_1)| \cdot |C_R(y_2)| \cdot |C_R(z)|.$$

Now $C_R(z) \leq C_G(v) \cap C_G(z) = C_H(v) = T$. As $|T| = 2^6$, we have $C_R(z) = 1$. Since R is normal in U and $e \in U$, it follows that $C_R(y_1) = C_R(y_2) = 1$ and $R = 1$.

(c) As $U = C_G(v) = \langle v \rangle \times J$, it follows from (a) and (b) that $O(J) = 1$ and J does not have a normal subgroup of odd index. Using MAGMA again, it can be checked that $C_W(x) = \langle x \rangle$ for each element x of order 8 of the Sylow 2-subgroup W of J , and W is not isomorphic to the wreath product of a cyclic group of order 4 by a cyclic group of order 2. Hence, $J > O^2(J)$ by Theorem 1 of Fong [5, p. 65]. In particular, J is not perfect and J/J' is a 2-group.

We claim that $\text{Soc}(J)$ is simple non-abelian. Let $M \neq 1$ be a minimal normal subgroup of J . Suppose that M is solvable. Then $M' = 1$, and M is a 2-group. Hence, M is a normal elementary abelian subgroup of W . By means of MAGMA, we see that the only such subgroups of W are the Klein four group V and $\langle z \rangle$. As $z^e = y_2 \notin V$ and $V \cap V^e = \langle zt \rangle$, we get $M = 1$, a contradiction. Since $Z(W)$ has order 2, W is not a direct product. By MAGMA, W has no abelian normal subgroup of index 2. Thus, $M := \text{Soc}(J)$ is simple. In particular, $C_J(M) = 1$ and $J \geq \text{Aut}(M)$. Since M is perfect, $M \leq J'$, hence its Sylow 2-subgroup W_1 has order at most 16.

Now $C_M(z) \leq M \cap C_G(z) \leq C_H(v) = \langle v \rangle \times W$. So $C_M(z) \leq W \cap M = W_1$.

If $|W_1| = 4$, then $C_M(W_1) = W_1$. By Burnside’s Theorem, $N_M(W_1) > W_1$, hence acts transitively on the three involutions of W_1 . Thus, $C_M(a) = W_1$ for any $a \in W_1^\#$. By [3, p. 92], $M \cong A_5$ and $J \leq S_5$, against $|W| = 32$.

Suppose that $|W_1| = 16$, then $W_1 \in \text{Syl}_2(J')$. In particular, since J has no normal subgroup of odd index and J/J' is abelian, we get $|J : J'| = 2$. Therefore, $M = O^{2'}(J')$, hence contains all 2-elements in J' . Since $B = \langle y_1, y_2, y_3, e \rangle \cong S_4$ is a subgroup of J , and $B' = \langle y_1, y_2, e \rangle \cong A_4 \leq J'$, it follows that $\langle y_1, y_2, e \rangle \leq M$. Now W' is cyclic of order 4, and MAGMA shows that $I = \langle W', y_2 \rangle$ is a dihedral group of order 8 and we may assume up to conjugation that it is contained in W_1 . From $W' \leq I$, it follows that $I \trianglelefteq W$. Now W/I has order 4, hence there exist three subgroups X_i ($i = 1, 2, 3$) of W such that $|X_i| = 16$ and $I \leq X_i$. They are semidihedral, dihedral, and a direct product of a cyclic group of order 2 and a dihedral subgroup of order 8, respectively. If $W_1 \cong X_3$, then M is not simple by Proposition 2.88 of Gorenstein [8, p. 124], a contradiction. If $W_1 \cong X_1$, then M has a unique conjugacy class of involutions z by Proposition 1 of [1, p. 10], and $|C_G(z)|$ is divisible by an odd prime p by the first main theorem of [1, p. 3], where $q = p^m$ is the order of the finite field over which $\text{PSL}_3(q)$ or $\text{U}_3(q)$ is defined. However, $z = y_1 y_2$ is in the center of W_1 , and $C_M(z) = W_1$ has order 16, a contradiction. Finally, suppose that $W_1 \cong X_2$. Then $M \cong \text{PSL}_2(q)$ for some odd prime power $q > 3$, or A_7 by the Gorenstein–Walter Theorem [7, p. 462]. If $M \cong A_7$, then $|C_M(z)| = 24$, which is a contradiction. In the first case, $|W_1| = |C_M(z)| \in \{q + 1, q - 1\}$. Hence, $q = 17$, $M \cong \text{PSL}_2(17)$, and $J \cong \text{Aut}(M) = \text{PGL}_2(17)$. Another application of MAGMA now yields that the Sylow 2-subgroups of $\text{PGL}_2(17)$ are not isomorphic to the Sylow 2-subgroup W of J . Therefore, $|W_1| = 8$, and $W_1 = I$ is a dihedral group of order 8. By the Gorenstein–Walter Theorem [7, p. 242], $M \cong A_d$, $d = 6, 7$. By the previous remarks, $J \leq \text{Aut}(M)$. As $\text{Aut}(A_7) \cong S_7$ has order not divisible by 32, $M \cong A_6$ and $J \cong \text{Aut}(A_6)$, which completes the proof of (c).

(e) It can be easily checked by means of MAGMA.

(f) A system of representatives u_i of the conjugacy classes u_i^U of U has been calculated by means of Kratzer’s programs [15].

(g) It follows immediately from the power map information given in Appendix A.4.

(h) Since $T = C_H(v) = H \cap U$ is a Sylow 2-subgroup of U , the fusion of the conjugacy classes of 2-elements of T in U is uniquely determined by Sylow’s Theorem. Using the faithful permutation representations of H and U determined in Lemma 2.1(m) and assertion (e), the fusion patterns of the conjugacy classes of T in H and U are calculated, respectively. The table is then determined by the methods described in the proof of Proposition 3.4(c). □

Definition 3.6. Let G be a finite group. A subgroup U of G is said to be *strongly embedded* in G if the following two conditions are satisfied: (1) U is a proper subgroup of even order. (2) For any element $x \in G \setminus U$, the order $|U \cap U^x|$ is odd.

Proposition 3.7. Let G be a finite simple group of HS-type having a 2-central involution z with centralizer $H = C_G(z)$. Let $A = \langle z, t, u \rangle$ be the uniquely determined

maximal elementary abelian normal subgroup of the fixed Sylow 2-subgroup S of G defined in Lemma 2.1, and let $E = N_G(A)$. Then the following assertions hold:

- (a) $|G| = 2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$.
- (b) $G = \langle H, E \rangle$.

Proof. (a) By Propositions 3.2 and 3.5, each simple group G of HS-type has two conjugacy classes of involutions z^G and v^G such that

$$|H| = |C_G(z)| = 2^9 \cdot 3 \cdot 5 \quad \text{and} \quad |U| = |C_G(v)| = 2^6 \cdot 3^2 \cdot 5.$$

Using the faithful permutation representations of H and of U described in Lemma 2.1 and Proposition 3.5, respectively, and the fusion patterns of the conjugacy classes of H and U in G given in Propositions 3.2 and 3.5, an application of MAGMA yields that

$$\begin{aligned} r(z, w, z) &= |\{(x, y) \in (z^G \cap H) \times (w^G \cap H) \mid z \in \langle xy \rangle\}| = 10720, \\ r(z, w, w) &= |\{(x, y) \in (z^G \cap U) \times (w^G \cap U) \mid z \in \langle xy \rangle\}| = 1755. \end{aligned}$$

Now Thompson’s group order formula [22, p. 198] states that

$$|G| = r(z, w, z) \cdot |C_G(w)| + r(z, w, w) \cdot |C_G(z)| = 2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11.$$

(b) By Lemma 2.1, the Sylow 2-subgroup S of G is neither abelian nor metabelian. Therefore, the Corollary of [21, p. 392] implies that the simple group G cannot have a strongly embedded subgroup. Thus, statement 4.3 of [21, p. 392] asserts that

$$G = \langle H, N_G(S), C_G(t) \mid t \in I(H) \rangle,$$

where $I(H)$ denotes the set of all involutions t of H . Since $Z(S) = \langle z \rangle$ by Lemma 2.1, we have $N_G(S) \leq H = C_G(z)$ because $Z(S)$ is a characteristic subgroup of S . By Proposition 3.4, G has two conjugacy classes of involutions. They are represented by z and v . Furthermore, by the proof of Proposition 3.4, for each $t \in I(H)$, there are elements $h_2, \dots, h_k \in H$ and $e_1, e_2, \dots, e_k \in E$ such that $t^{e_1 h_2 e_2 \dots h_k e_k} \in \{z, v\}$. Let $x = e_1 h_1 e_2 h_2 \dots e_k h_k$. Then $x \in \langle H, E \rangle$ and either $[C_G(t)]^x = C_G(t^x) = C_G(z) = H$ or $[C_G(t)]^x = C_G(t^x) = C_G(v) = U$. Hence, $G = \langle H, E, U \rangle$.

By Proposition 3.5, $U = \langle v \rangle \times \langle y_1, y_2, y_3, y_4, e \rangle$, where $y_i \in D = \langle r, s \rangle \leq H$ and $v, e^{g^2} = sr s^3 c^{-1} \in H, g$ as in Proposition 3.2. Hence, $U \leq \langle H, E \rangle$, which completes the proof. □

Proposition 3.8. *Let G be a finite simple group of HS-type having a 2-central involution z with centralizer $H = C_G(z) = \langle r, s, c \rangle$ defined in Lemma 2.1. Let $A = \langle z, t, u \rangle$ be the unique maximal elementary abelian normal subgroup of order 8 in the fixed Sylow 2-subgroup S of H , and let g be the element of order 3 in G of Proposition 3.2 satisfying $z^g = t, u^g = u$, and $N_G(A) = \langle r, s, g \rangle$. Let $v = (rs^{-1})^4 s^2$, and $U = C_G(v) = \langle v, y_1, y_3, y_4, e \rangle$ be as in Proposition 3.5, and $k = y_4 y_2 (e^{-1} y_4)^2 e y_4$. Then the following statements hold:*

- (a) G has a unique conjugacy class 3_A of elements of order 3. In particular, r, g, e are conjugate in G .

- (b) $C_G(e)/\langle e \rangle = K_1 = \langle a, k \rangle \cong S_5$ has the following set $\mathcal{R}(K_1)$ of defining relations:

$$k^3 = 1 = a^4, \quad k^{a^2} = k^2 \quad \text{and} \quad (ak)^6 = 1,$$

where $a = [(cr^2)^3]^{h_1^{-1}g^2}$ for $h_1 = rs^{-1}cs^2 \in H$ of order 6.

- (c) $N_0 = N_G(3_A) = N_G(e) = \langle C_G(e), y_3 \rangle = \langle a, e, k, y_3 \rangle \cong S_3 \times S_5$.
 (d) G has five e -special conjugacy classes represented by the following classes of $N_G(e)$ given in Appendix A.5: $3_a, 6_b, 6_c, 12_a, 15_b$.
 (e) A system of representatives x_i of the twenty one conjugacy classes $x_i^{N_0}$ of $N_0 = N_G(3_A) = \langle a, e, y_3, k \rangle$ and their centralizer orders $|C_{N_0}(x_i)|$ are given in Appendix A.5.
 (f) The fusion of the conjugacy classes of $N_0 = N_G(3_A)$ into the conjugacy classes of G is given by the following table:

G	2_A		2_B			3_A	4_A	4_C	6_B
N_0	2_c	2_d	2_a	2_b	2_e	3_b	4_a	4_b	6_e
H	2_b	2_d	2_c	2_e		3_a	4_d	4_g	6_b

Proof. (a) By Proposition 3.7, $|G| = 2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$, hence U contains a Sylow 3-subgroup of G . Using the faithful permutation representation of U defined in Proposition 3.5 and MAGMA, it can be checked that $C_U(e) = \langle e \rangle \times \langle v \rangle \times W_1$, where $W_1 = \langle k, p \rangle$ with $k = y_4 y_2 (e^{-1} y_4)^2 e y_4$ and $p = tz$ satisfying the relations $k^3 = 1 = p^2$ and $k^p = k^2$. By MAGMA, we got only one conjugacy class of three elements in U , hence in G .

(b) Using the faithful permutation representation of H defined in Lemma 2.1 and MAGMA, it follows that $C_H(r) = \langle r \rangle \times L_1$, where $L_1 = \langle z_1, z_2 \rangle$ satisfies the relations $z_1^2 = 1 = z_2^4$ and $z_2^{z_1} = z_2^3$ for $z_1 = cs^{-1}csr^{-1}s^{-1}c$ and $z_2 = (cr^2)^3$. Furthermore, the center of the dihedral group L_1 is z . Hence, L_1 is a Sylow 2-subgroup of $C_G(r)$. By (a), $e \in U = C_G(v)$ and r are conjugate in G . Furthermore, $e^g = srs^3c^{-1} \in H$ by Proposition 3.5(d). Another application of MAGMA now yields that $e^{gh_1} = r$ for $h_1 = rs^{-1}cs^{-2} \in H$ of order 6. Thus, $L_1^{h_1^{-1}g^2}$ is a Sylow 2-subgroup of $C_1 = C_G(e)$ by (a). Since $C_U(e) = \langle e \rangle \times \langle v \rangle \times \langle k, p \rangle$, by Gaschütz’s Theorem, $\langle e \rangle$ has a complement K_1 in $C_G(e)$ and, up to isomorphism, $K_1 \geq \langle L_2, v, k \rangle$. Set $a = z_2^{h_1^{-1}g^2}$.

As $z, z_1, z_2 \in L_1$, the elements $z^{h_1^{-1}g^2} = tz, z_1^{h_1^{-1}g^2} = v$ and $z_2^{h_1^{-1}g^2} = a$ belong to the Sylow 2-subgroup $L_2 = L_1^{h_1^{-1}g^2}$ of K_1 , and k does not normalize the Klein four subgroup $\langle v, z \rangle$ of L_2 . Hence, $K_1/O(K_1)$ contains a normal subgroup $K_2 \cong \text{PSL}_2(q)$ by the Gorenstein–Walter Theorem. Now for any involution $z_1 \in K_2$, $|C_{K_2}(z_1)| \in \{4, 8\} \cap \{q \pm 1\}$. If $q - 1 = 8$, then $K_2 = \text{PSL}_2(9)$ has an elementary abelian Sylow 3-subgroup of order 9, but 3 divides $|K_1|$ to the first power only. If $q + 1 = 8$, then in $K_2 = \text{PSL}_2(7) \cong \text{PSL}_3(2)$, the involutions z and v are conjugated, another contradiction. Since k does not normalize $\langle v, z \rangle$, K_2 is not isomorphic to a symmetric group S_4 . Hence, $q > 3$, and $q = 5$. Therefore, $K_2 \cong \text{PSL}_2(5) \cong A_5$, and K_2 contains an alternating group A_4 as subgroup. Applying then the Brauer–Wielandt formula as in Proposition 3.5, it follows that $O(K_1) = 1$. Hence, $C_G(e) = \langle e \rangle \times K_1$ and $K_1 = \langle a, k \rangle \cong S_5$, where $a^4 = 1 = k^3$, $(ak)^6 = 1$ and $k^{a^2} = k^2$.

(c) Using the fact that $e^{y_3} = e^2$, it follows from (b) that $N_0 = N_G(e) = C_G(e)\langle y_3 \rangle \cong S_3 \times S_5$.

(d) The 3_A -special conjugacy classes of G are easily read off from Appendix A.5.

(e) The representatives of the conjugacy classes of $N_0 = N_G(e) = \langle e, k, a, y_3 \rangle$ have been calculated with Kratzer's program [15].

(f) Observe that $|N_G(r) \cap H| = 48$. Hence, this intersection contains the Sylow 2-subgroup of $N_0 = N_G(e)$ up to conjugation in G . Now apply the proof of Proposition 3.5(f) and the fusion follows. \square

Lemma 3.9. *Let G be a finite simple group of HS-type having a 2-central involution z with centralizer $H = C_G(z)$. Then the following statements hold:*

- (a) G has a unique conjugacy class 7_A of elements s_1 of order 7, $C_G(s_1) = \langle s_1 \rangle$, and $N_G(s_1)/C_G(s_1)$ is a cyclic group of order 6.
- (b) A system of representatives y_i and the corresponding centralizer orders $|C_{N_3}(y_i)|$ of the seven conjugacy classes of $N_3 = N_G(7_A) = \langle s_1, t_1 \rangle$ is given in the following table:

i	y_i	$ C_{N_3}(y_i) $	2P	3P	7P
1_a	1	$2 \cdot 3 \cdot 7$	1_a	1_a	1_a
2_a	$(t_1)^3$	$2 \cdot 3$	1_a	2_a	2_a
3_a	$(t_1)^2$	$2 \cdot 3$	3_b	1_a	3_a
3_b	$(t_1)^4$	$2 \cdot 3$	3_a	1_a	3_b
6_a	t_1	$2 \cdot 3$	3_a	2_a	6_a
6_b	$(t_1)^5$	$2 \cdot 3$	3_b	2_a	6_b
7_a	s_1	7	7_a	7_a	1_a

(c) The character table of $N_3 = N_G(7_A)$ is stated in Appendix B.7.

(d) G has two conjugacy classes 11_A and 11_B of elements q_1 and q_2 of order 11, respectively. Furthermore, $C_G(q_i) = \langle q_i \rangle$ and $N_G(q_i)/C_G(q_i)$ is a cyclic group of order 5 for $i = 1, 2$.

Proof. Let G_p denote a Sylow p -subgroup of G . By Proposition 3.7,

$$|G| = 2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11,$$

and it follows from Propositions 3.5(e) and 3.8(c) that $|C_G(G_{11})| \mid 5^3 \cdot 7 \cdot 11$. On the other hand, $|N_G(G_{11}) : C_G(G_{11})| \mid 10$. So $|N_G(G_{11})| = 2^a \cdot 5^b \cdot 7^c \cdot 11$, where $0 \leq a, c \leq 1$ and $0 \leq b \leq 4$. Now $|G : N_G(G_{11})| \equiv 1 \pmod{11}$ forces $a = c = 0$ and $b = 1$. By Burnside's Theorem [7, p. 252], $N_G(G_{11}) = C_G(G_{11})$ would imply that G is 11-nilpotent, a contradiction, because G is simple. Thus, $|N_G(G_{11})| = 5 \cdot 11$ and $|C_G(G_{11})| = 11$.

Similarly, $|C_G(G_7)| \mid 5^3 \cdot 7 \cdot 11$. By Proposition 3.2(f), E possesses two conjugacy classes of elements of order 7. Hence, $|N_G(G_7) : C_G(G_7)| \in \{3, 6\}$. Thus, $|N_G(G_7)| = i \cdot 5^a \cdot 7 \cdot 11$, where $i = 3, 6$, $0 \leq a \leq 3$ and $0 \leq b \leq 1$. The only triples (i, a, b) compatible with Sylow's Theorem are $(6, 0, 0)$ and $(3, 2, 1)$ as has been checked computationally. But the second alternative would imply that a 7-element commutes with an 11-element. So $|N_G(G_7)| = 6 \cdot 7$ and $|C_G(G_7)| = 7$. The remaining assertions are easily proved by means of MAGMA. \square

Proposition 3.10. *Let G be a finite group of HS-type having a 2-central involution z with centralizer $H = C_G(z) = \langle r, s, c \rangle$. Then the following assertions hold:*

- (a) $C_H(c) = \langle c \rangle \times \langle z_2 \rangle$, where $z_2 = (rs^2)^3$ has order 4, and $C_G(c)$ has a cyclic Sylow 2-subgroup $\langle z_2 \rangle$.
- (b) $N_H(c) = \langle c \rangle : S_1$, where $S_1 = \langle a_1, b_1 \rangle$ is a Sylow 2-subgroup of order 16, and $a_1 = s^{-1}(cs)^2r$, $b_1 = sr^{-1}csc$ satisfy the following set $\mathcal{R}(S_1)$ of defining relations:

$$a_1^8 = 1 = b_1^2, \quad a_1^{b_1} = a_1^5.$$

- (c) $\langle c \rangle$ is the center of a Sylow 5-subgroup R of G which is contained in $N_G(c)$, R is extraspecial of order 125 and exponent 5. It is generated by two elements $x, y \in P$ with the following set $\mathcal{R}(R)$ of defining relations:

$$[x, y] = c^2, \quad x^5 = y^5 = 1 \quad \text{and} \quad [c, y] = [c, x] = 1.$$

- (d) $N_1 = N_G(c) = R : S_1 = N_G(R) = \langle a_1, b_1, x, y \rangle$ has the following set $\mathcal{R}(N_1)$ of defining relations:

$$\mathcal{R}(S), \mathcal{R}(R), x^{a_1} = y, y^{a_1} = x^2, x^{b_1} = x^{-1} \text{ and } y^{b_1} = y.$$

- (e) A system of representatives n_i and corresponding centralizer orders $|C_{N_1}(n_i)|$ of the 19 conjugacy classes $n_i^{N_1}$ of the normalizer $N_1 = N_G(5_A)$ are given in Appendix A.6.
- (f) The character table of $N_1 = N_G(5_A)$ is given in Appendix B.6.
- (g) G has at least two conjugacy classes 5_A and 5_B of elements of order 5 represented by c and x , respectively.
- (h) $N_2 = N_G(5_B) = N_G(c_2) \cong \langle c_2 \rangle : \langle f \rangle \times A_5 = \langle c_2, f, w, v \rangle$, where $c_2 = (y_4e)^2$ and $f = y_1y_4ey_4y_3e^2y_4ey_4$ have orders 5 and 4, respectively, and $A_5 = \langle w, v \rangle$ has defining relations $v^2 = w^5 = (wv)^3 = 1$.
- (i) A system of representatives m_i and corresponding centralizer order $|C_{N_2}(m_i)|$ of the twenty five conjugacy classes $m_i^{N_2}$ of the normalizer $N_2 = N_G(5_B)$ are given in Appendix A.7.
- (j) The character table of $N_2 = N_G(5_B)$ is given in Appendix B.8.
- (k) Using the notations of the conjugacy classes of $H = C_G(z)$, $U = C_G(v)$, $N_1 = N_G(5_A)$ and $N_2 = N_G(5_B)$ as in Appendices A.1, A.4, A.6 and A.7, respectively, their fusion patterns into the conjugacy classes of G are given by the following tables:

G	2_A	2_B	4_A	4_C		8_A		8_C		10_A	20_A	20_B
N_1	2_a	2_b	4_a	4_b	4_c	8_a	8_b	8_c	8_d	10_a	20_b	20_a
H	2_a	2_e	4_a	4_e		8_a		8_e		10_a	20_a	20_b

G	2_A	2_B			4_A		4_C			5_B	10_B			
N_2	2_a	2_b	2_c		4_a	4_b	4_c	4_d		5_a	10_c			
U	2_f	2_a	2_d	2_e	2_g	4_c		4_a	4_e	4_f	5_a	10_a	10_b	10_c

Proof. (a) As $c \in H$ has order 5, we calculate $C_H(c)$. By MAGMA, it is a direct product $\langle c \rangle \times \langle z_2 \rangle$, where $\langle z_2 \rangle = Z(O_2(H)) = Z(H')$ has order 4, and $z_2^2 = z$. As seen before, this implies that $L = \langle z_2 \rangle \in \text{Syl}_2(C_G(c))$.

(b) This assertion has been checked computationally by MAGMA.

(c) From (a) and Burnside's Theorem, it follows that $M = C_G(c)$ has a normal 2-complement $R = O(M)$. Using MAGMA again, we see that $V_1 = \langle z, q_1 \rangle \leq N_H(c)$ is a Klein four group, where $q_1 = sr^{-1}csc$ is conjugate to q_1z in $N_H(c)$. By Brauer–Wielandt's Formula [10],

$$|R| \cdot |C_R(V_1)|^2 = |C_R(z)| \cdot |C_R(q_1)| \cdot |C_R(q_1z)|.$$

Now $C_R(z) = \langle c \rangle$ and $C_R(V_1) = 1$ since z is the only involution in $C_H(c)$. On the other hand, $25 \mid |R|$. So $5 \mid |C_R(q_1)| = |C_R(zq_1)|$ and $125 \mid |R|$. By Proposition 3.7 and Lemma 3.9, R is a $\{3, 5\}$ -group. If 3 divides $|R|$, then $9 \mid |R|$ too. Since $C_G(e) \cong \mathbb{Z}_3 \times S_5$, this would force the existence of an element of order 15 in S_5 , a contradiction. Thus, $|R| = 125$, $R \in \text{Syl}_5(G)$, and $|C_G(c)| = 2^2 \cdot 5^3$. Since $2^4 \mid |N_H(c)|$ and $|N_G(c) : C_G(c)| \mid 4$, we have $|N_G(c)| = 2^4 \cdot 5^3$. Before we can complete the proof of (c), we first have to prove (h).

(h) Observe that $c_2 = (y_4e)^2 \in U$ has order 5. Applying MAGMA, we see that $C_U(c_2) = \langle c_2 \rangle \times \langle v, v_2 \rangle$, where $v_2 = y_1ey_1y_4(ey_4)^2e^2$ is an involution. Furthermore, $N_U(c_2) = (\langle c_2 \rangle : \langle f \rangle) \times \langle v, v_2 \rangle$, where $f = y_1y_4ey_4y_3e^2y_4ey_4$ has order 4, satisfies $c_2^f = c_2^2$, and $T = \langle v, v_2 \rangle$ is a Klein four group. Suppose that $z^G \cap T$ contains an involution i . Then $z = i^x$ for some $x \in G$, and $c_2^{xh} \in H = C_G(z)$. Therefore, $c = c_2^{xh}$ for some $h \in H$ by Lemma 2.1(n). Hence, $T^{xh} \leq C_G(c)$, a contradiction to (a). Hence, all involutions of T are conjugate to v in G , i.e., $T^\# = T \cap v^G$.

Let $X = C_G(c_2)$. We now claim that T is a Sylow 2-subgroup of X . Suppose that T is a proper normal subgroup of a 2-subgroup Y of X . Then $Z(Y) \cap T \neq 1$. Let $i \in Z(Y) \cap T^\#$. Then $v = i^x$, $c_2^x \in U$, and so $Y^x \leq U$.

Now there is $u \in U$ such that $c_2^x = c_2^u$, so $Y^x \leq C_U(c_2^x) = C_U(c_2)^u$ and $Y^x \leq T^u$, forcing $T = Y \in \text{Syl}_2(X)$.

Set $\bar{X} = X/\langle c_2 \rangle$. Now $C_X(T) = C_G(v) \cap C_G(v_2) \cap X = C_U(v_2) \cap X$. Since $C_U(v_2)$ is a 2-group, we get that $C_{\bar{X}}(T) = T$ is a self-centralizing subgroup of \bar{X} .

Set $P = O(X)$. Using the structure of subgroups of odd order in both H and $C_G(v)$, an easy application of the Brauer–Wielandt fixed-point formula for the action of T on P now yields $P = \langle c_2 \rangle$. Let M be a minimal normal subgroup of \bar{X} . Since 5 divides $|\bar{X}|$, \bar{X} is not a 2-group. In particular, as $C_{\bar{X}}(T) = T$, the group T is not normal in \bar{X} . Thus, M is a simple group and $T \in \text{Syl}_2(M)$. By Theorem 15.2.1 in [7, p. 421], $M \cong \text{PSL}_2(q)$, $q \equiv \pm 3 \pmod 8$. Moreover, $|M| \mid 2^2 \cdot 3^2 \cdot 5^3$, thus $q = 5$ and $M \cong A_5$. Since $|\bar{X} : M|$ is odd, we get $C_{\bar{X}}(M) = 1$ and $\bar{X} = M$. Therefore, X is an extension of \mathbb{Z}_5 by A_5 . Since the Schur multiplier of A_5 is \mathbb{Z}_2 , we obtain $C_G(c_2) \cong \langle c_2 \rangle \times A_5$. Since $N_U(c_2)/C_U(c_2) \cong \mathbb{Z}_4$, we obtain $N_G(c_2) \cong (\langle c_2 \rangle \times A_5) : \mathbb{Z}_4$. MAGMA shows that $N_G(c_2) \cong (\mathbb{Z}_5 : \mathbb{Z}_4) \times A_5$. Clearly, the involution $v \in A_5$. The existing additional generator w of A_5 can be chosen to be of order 5 satisfying the given relations with v .

(d) In particular, $R \in \text{Syl}_5(G)$ is not abelian, $c \in Z(R)$, and $N_G(R) \leq N_G(c)$. On the other hand, $R = O(C_G(c))$ is characteristic in $C_G(c)$, hence normal in

$N_G(c)$. Thus, $N_G(c) = N_G(R)$. Moreover, R is extraspecial of order 125. Since z_2 acts fixed-point-freely on $R/\langle c \rangle$, it has exponent 5. Since $|N_H(c)|_2 = 16$, H contains a Sylow 2-subgroup S_2 of $N_G(R)$, which is generated by a_1 and b_1 subject to the relations $a_1^8 = b_1^2 = a_1^3 a_1^{b_1} = 1$. Moreover, $l = a_1^6 b_1$ generates $C_H(c)$. Now $\langle a, b \rangle$ acts on $\overline{R} = R/\langle c \rangle$, a 2-dimensional vector space over $F = GF(5)$. Up to conjugation in $GL_2(5)$, we may assume that a_1 and b_1 act as the matrices $\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, respectively.

Thus, $N_G(R)$ may be obtained as the finitely presented group generated by x, y, a_1, b_1 subject to the relations $x^5 = y^5 = [x, y, x] = [x, y, y] = a_1^8 = b_1^2 = a_1^3 a_1^{b_1} = y^4 x^{a_1} = x^3 y^{a_1} = x x^{b_1} = y^4 y^{b_1} = 1$. MAGMA shows that only twenty five 4-tuples $(\alpha, \beta, \gamma, \delta)$ over \mathbb{Z}_5 allow to lift the action of $\langle a_1, b_1 \rangle$ on R , where $x^{a_1} = y c^\alpha, \dots, y^{b_1} = y c^\delta$, but they all give rise to isomorphic groups. Again MAGMA proves that $N_G(c)$ contains exactly three conjugacy classes of 5-elements represented by c, x and xy . Clearly, $25 \mid |C_G(xy)|$. We will prove in the next proposition that $|C_G(xy)| = 25$, hence $C_G(xy) \leq C_G(c)$. Now $(xy)^{y a_1^2} = (xy)^2$, so $N_G(xy) = C_G(xy) : \langle y a_1^2 \rangle \leq N_G(c)$. Observe that the claims in part (c) have been proved by now as well.

(i) The representatives of the conjugacy classes of $N_2 = N_G(5_B)$ have been calculated with Kratzer’s program [15].

(j) and (k) The fusion patterns of the conjugacy classes of $N_1 = N_G(5_A)$ and $N_2 = N_G(5_B)$ have been determined by means of the faithful permutation representations of N_1 and N_2 and MAGMA, respectively. The character table of N_2 has been calculated similarly. □

Proposition 3.11. *Let G be a finite simple group of HS-type having a 2-central involution z with centralizer $H = C_G(z)$. The character tables of the local subgroups $C_G(2_A)$, $C_G(2_B)$, $N_G(3_A)$, $N_G(5_A)$, $N_G(5_B)$ and $N_G(7_A)$ are given in Appendix B. Keeping their notations for the conjugacy classes of these local subgroups of G , the following statements hold:*

- (a) G has twenty four conjugacy classes x^G with centralizer orders $|C_G(x)|$ given in the table below.
- (b) The fusion of the conjugacy classes of the six local subgroups of G in the conjugacy classes of G is described by the following table:

x	$ C_G(x) $	$C_G(2_A)$	$C_G(2_B)$	$N_G(3_A)$	$N_G(5_A)$	$N_G(5_B)$	$N_G(7_A)$
1_A	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$	a	a	a	a	a	a
2_A	$2^9 \cdot 3 \cdot 5$	a, b, d	b, f	c, d	a	a	
2_B	$2^6 \cdot 3^2 \cdot 5$	c, e	a, c, d, e, g	a, b, e	b	b, c	a
3_A	$2^3 \cdot 3^2 \cdot 5$	a	a	a, b, c		a	a, b
4_A	$2^8 \cdot 3 \cdot 5$	a, d	c	a, b	a	a, b	
4_B	2^8	b, c, f, h	d				
4_C	2^6	e, g	a, b, e, f		b, c	c, d	
5_A	$2^2 \cdot 5^3$	a			a	b, c	
5_B	$2^2 \cdot 3 \cdot 5^2$		a	a	b	a	
5_C	5^2			c	c	d, e	
6_A	$2^2 \cdot 3^2$	a, c	b	a, b, d, e		a	
6_B	$2^3 \cdot 3$	b	a, c	c, f			a, b
7_A	7						a
8_A	2^4	a	c, d		a, b		
8_B	2^4	b, c, d					
8_C	2^4	e	a, b		c, d		
10_A	$2^2 \cdot 5$	a			a	a, b	
10_B	$2^2 \cdot 5$		a, b, c	a	b	c	
11_A	11						
11_B	11						
12_A	$2^2 \cdot 3$	a		a		a, b	
15_A	$3 \cdot 5$			a		a	
20_A	$2^2 \cdot 5$	a			b	a, b, c, d	
20_B	$2^2 \cdot 5$	b			a		

The orders of the representatives of the conjugacy classes of the six local subgroups of G are omitted because they can be read off from the first column of the table.

Proof. (a) By Proposition 3.4, each finite simple group G of HS-type has two conjugacy classes of involutions 2_A and 2_B represented by the involutions z and v , respectively. Therefore, the representatives of the z - and v -special conjugacy classes of even order given in Propositions 3.2(f) and 3.5(e), respectively, yield together a complete system of representatives g_i of all conjugacy classes of G having even orders. The orders $|C_G(g_i)|$ of their centralizers can be read off from Appendices A.1 and A.4.

By Proposition 3.8, G has only one conjugacy class 3_A of elements of order 3, $|C_G(3_A)| = 2^3 \cdot 3^2 \cdot 5$, and one odd order 3_A -special conjugacy class 15_A with $|C_G(15_A)| = 15$. The conjugacy classes 7_A , 11_A and 11_B of G of elements of orders 7 and 11, respectively, have been classified in Lemma 3.9, which also states that the subgroups $\langle 7_A \rangle$, $\langle 11_A \rangle$ and $\langle 11_B \rangle$ are self-centralizing in G . By Proposition 3.10, we know that only one further class of 5-elements is left. Summing up the lengths of the twenty three classes of G obtained so far and comparing the sum with $|G| = 2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$, we get $|C_G(c_3)| = 25$ because $N_G(5_A)$ has three conjugacy classes of elements of order 5 by Appendix A.6. Hence, G has twenty four conjugacy classes x^G with corresponding centralizer orders $|C_G(x)|$ as given in the second column of the table above.

(b) The fusion of the conjugacy classes of $C_G(2_A)$ in G is given in Proposition

3.4(c) except for the ones of $6_a, 6_b$ and 6_c . By definition, $6_A = 6_a$. Since G has only one conjugacy class of elements of order 3, the power maps $(6_b)^3$ and $(6_c)^3$ given in Appendix A.1 and the fusion of the involutions of H in G prove that 6_a and 6_c fuse to 6_A , and 6_b fuses to 6_B .

The fusion of the conjugacy classes of $N_G(3_A)$ in G follows similarly from Proposition 3.8, and the ones of $N_G(5_A)$ and $N_G(5_B)$ from Proposition 3.10. As the involution v centralizes e of 3_A , it follows that the involution of $N_G(7_A)$ centralizing the elements of order 3 there is G -conjugate to v . Thus, the fusion of the classes of $N_G(7_A)$ in G is clear by the table in Lemma 3.9(b). \square

4 Existence Proof of HS Inside $GL_{22}(11)$

In this section, we apply Algorithm 4.6 of [17] to give a new existence proof for a finite simple group G of HS-type. Throughout, $H = \langle r, s, c \rangle$ is the group of even order defined in Lemma 2.1, and $A = \langle z, t, u \rangle$ is the unique maximal elementary abelian normal subgroup of the fixed Sylow 2-subgroup S described there. Then $D = N_H(A) = \langle r, s \rangle$, and $C = C_H(A)$ is the unique homocyclic abelian subgroup of order 64 of S . Furthermore, D splits over C , and $\Delta = D/C \cong S_4$ by Lemma 2.1. By Proposition 3.2, Δ is a uniquely determined maximal subgroup of $\Phi \cong L_3(2)$ with $|\Phi : \Delta| = 7$, and there is a uniquely determined extension $1 \rightarrow C \rightarrow E \rightarrow \Phi \rightarrow 1$, where $E = C : \Phi$ is generated by $D = \langle r, s \rangle$ and an element g of order 3 such that $\Phi = \langle r, g \rangle$ and $E = \langle r, s, g \rangle$ satisfy the set $\mathcal{R}(E)$ of relations given in Proposition 3.2(f).

Let $\text{mfchar}(X)$ be the set of all multiplicity-free faithful characters of the group $X \in \{H, D, E\}$. By means of Kratzer’s algorithm [15], we calculate the finite set $\Pi = \{(\eta, \tau) \in \text{mfchar}(H) \times \text{mfchar}(E) \mid \eta|_D = \tau|_D\}$ of *compatible pairs*. For each pair $(\eta, \tau) \in \Pi$, the positive integer $\eta(1) = \tau(1)$ is called the *degree of the compatible pair* (η, τ) . We denote with boldface indices the faithful constituents.

Lemma 4.1. *There is a unique compatible pair (η, τ) of multiplicity-free faithful characters $\eta \in \text{mfchar}(H)$ and $\tau \in \text{mfchar}(E)$ with minimal degree $\eta(1) = 22 = \tau(1)$, namely $(\eta, \tau) = (\eta_{\mathbf{3}} + \eta_{\mathbf{12}} + \eta_{\mathbf{15}}, \tau_{\mathbf{1}} + \tau_{\mathbf{6}} + \tau_{\mathbf{10}})$. The common restriction to $D \cong 2^{3+3} : S_4$ is $\eta|_D = \delta_{\mathbf{1}} + \delta_{\mathbf{5}} + \delta_{\mathbf{11}} + \delta_{\mathbf{14}} + \delta_{\mathbf{22}}$.*

Proof. All assertions follow immediately from Lemma 2.1 and Proposition 3.2(g) and Kratzer’s algorithm [15]. \square

Theorem 4.2. *Let $(\eta, \tau) \in \text{mfchar}(H) \times \text{mfchar}(E)$ be the unique compatible pair of degree 22 of the groups $H = \langle r, s, c \rangle$, $E = \langle r, s, g \rangle$ and $D = \langle r, s \rangle$. Let \mathfrak{V} and \mathfrak{W} be the up to isomorphism uniquely determined faithful semisimple multiplicity-free 22-dimensional modules of H and E over $F = GF(11)$ corresponding to η and τ , respectively. Let $\kappa_{\mathfrak{V}} : H \rightarrow GL_{22}(11)$ and $\kappa_{\mathfrak{W}} : E \rightarrow GL_{22}(11)$ be the representations of H and E afforded by the modules \mathfrak{V} and \mathfrak{W} , respectively. Let $\mathcal{R} = \kappa_{\mathfrak{V}}(r)$, $\mathcal{S} = \kappa_{\mathfrak{W}}(s)$, and $\mathcal{C} = \kappa_{\mathfrak{V}}(c) \in \kappa_{\mathfrak{V}}(H) \leq GL_{22}(11)$. Then the following assertions hold:*

- (a) $\mathfrak{V}|_D \cong \mathfrak{W}|_D$, and there is a transformation matrix $\mathcal{T} \in GL_{22}(11)$ such that $\mathcal{R} = \mathcal{T}^{-1}\kappa_{\mathfrak{W}}(r)\mathcal{T}$ and $\mathcal{S} = \mathcal{T}^{-1}\kappa_{\mathfrak{W}}(s)\mathcal{T}$.
- (b) Let $\mathcal{G} = \mathcal{T}^{-1}\kappa_{\mathfrak{W}}(g)\mathcal{T}$. Then $\mathfrak{G} := \langle \mathcal{R}, \mathcal{S}, \mathcal{C}, \mathcal{G} \rangle \leq GL_{22}(11)$ is a finite simple

Using the Meataxe algorithm of MAGMA over the prime field $F = GF(11)$, the corresponding irreducible FH - and FE -modules $\mathfrak{X}_3, \mathfrak{X}_{12}, \mathfrak{X}_{15}$ and $\mathfrak{Y}_1, \mathfrak{Y}_6, \mathfrak{Y}_{16}$ are extracted from the faithful FH - and FE -modules $F\mathfrak{X}$ and $F\mathfrak{Y}$, respectively. Then $\mathfrak{V} = \mathfrak{X}_3 \oplus \mathfrak{X}_{12} \oplus \mathfrak{X}_{15}$ and $\mathfrak{W} = \mathfrak{Y}_1 \oplus \mathfrak{Y}_6 \oplus \mathfrak{Y}_{16}$ are the faithful semisimple multiplicity-free FH - and FE -modules corresponding to the compatible pair (η, τ) .

As $\eta|_D = \tau|_D$, the restrictions $\mathfrak{V}|_D$ and $\mathfrak{W}|_D$ are isomorphic semisimple FD -modules. An application of MAGMA provides then the transformation matrix $T \in GL_{22}(11)$. Now Lemma 4.1 and Thompson’s Theorem [22, p. 148] assert that \mathfrak{V} extends to a uniquely determined irreducible 22-dimensional $F\mathfrak{G}$ -module. Identify H with its image $\kappa_{\mathfrak{V}}(H)$ in \mathfrak{G} . Since \mathfrak{V}_H decomposes into three irreducible FH -modules of dimension 4, 8, and 10, right multiplication with elements $g \in \mathfrak{G}$ on \mathfrak{V} induces a permutation action of \mathfrak{G} on \mathfrak{V} of degree 5775. Using MAGMA [2], it has been checked that $\kappa_{\mathfrak{V}}(H) = H$ is the stabilizer of this permutation module of \mathfrak{G} . Hence, $|\mathfrak{G}| = |H| \cdot 5775 = 2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$. Another application of this permutation representation and MAGMA [2] yields that \mathfrak{G} is simple, and $\mathcal{Z} = \mathcal{S}^4$ is a 2-central involution of \mathfrak{G} with $C_{\mathfrak{G}}(\mathcal{Z}) = \langle \mathcal{R}, \mathcal{S}, \mathcal{C} \rangle = \mathfrak{H} \cong H$.

The two assertions (d) and (e) follow from Kratzer’s programs [15] and MAGMA applied to the explicit permutation representation $(1_{\mathfrak{H}})^{\mathfrak{G}}$ constructed above. \square

5 Uniqueness of HS

In this section, we complete our proof of the Janko–Wong Theorem [14] stating that all finite simple groups G with given centralizer $H = C_G(z)$ of a 2-central involution z are isomorphic. In view of the first author’s uniqueness criterion [18] and the result of the previous sections, it now suffices to show the following proposition.

Proposition 5.1. *Each finite simple group G of HS-type has a unique irreducible character $\chi : G \rightarrow \mathbb{C}$ of degree $\chi(1) = 22$.*

Proof. Keep the notation of Proposition 3.11 for the conjugacy classes of G . Define the class function $\chi : g \rightarrow \chi(g) \in \mathbb{C}$ for all $g \in G$ by:

g	1_A	2_A	2_B	3_A	4_A	4_B	4_C	5_A	5_B	5_C	6_A	6_B
$\chi(g)$	22	6	-2	4	-6	2	2	-3	2	2	-2	0

g	7_A	8_A	8_B	8_C	10_A	10_B	11_A	11_B	12_A	15_A	20_A	20_B
$\chi(g)$	1	0	0	0	1	-2	0	0	0	-1	-1	-1

Let p be any prime divisor of $|G|$. A subgroup X of G is called p -elementary if X is a direct product of a p -subgroup Q and a cyclic group C . Let q be an element in the center $Z(Q)$ of Q having order p . Then $X \leq N_G(q)$. Propositions 3.7 and 3.11 assert that G has the following conjugacy classes of elements of prime order:

$$2_A, 2_B, 3_A, 5_A, 5_B, 5_C, 7_A, 11_A, 11_B.$$

Therefore, each p -elementary subgroup X is G -conjugate to a subgroup of one of the normalizers of these p -elements. As $|G| = 2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$ and the degree of χ is 22, Theorem 8.17 of Isaacs [13, p. 133] states that we need not restrict the class

function χ to the normalizer of the cyclic subgroups $\langle 11_A \rangle$ and $\langle 11_B \rangle$. Furthermore, by Proposition 3.10, we know that $N_G(5_C)$ is conjugate to a subgroup of $N_G(5_A)$ in G .

Therefore, by Theorem 8.4 of Isaacs [13, p.127], we have to show that $\chi|_Y$ is a character of the group Y , whenever

$$Y \in \mathfrak{Y} = \{H = C_G(2_A), U = C_G(2_B), N_G(3_A), N_G(5_A), N_G(5_B), N_G(7_A)\}.$$

In Proposition 3.11, the fusion of the conjugacy classes of these groups $Y \in \mathfrak{Y}$ into the conjugacy classes of G has been determined. Therefore, the inner products $(\chi, \lambda)_Y$ can be calculated for each $\lambda \in \text{Irr}_{\mathbb{C}}(Y)$ and each $Y \in \mathfrak{Y}$ by means of the character tables of the six groups Y . They are given in Appendix B. Let $Y_1 = N_G(3_A)$, $Y_2 = N_G(5_A)$, $Y_3 = N_G(5_B)$ and $Y_4 = N_G(7_A)$. Using MAGMA, it follows that

$$\begin{aligned} \chi|_H &= \chi_3 + \chi_{12} + \chi_{15}, \text{ where } \chi_3, \chi_{12}, \chi_{15} \in \text{Irr}_{\mathbb{C}}(H), \\ \chi|_U &= \chi_3 + \chi_4 + \chi_5 + \chi_{13} + \chi_{17}, \text{ where } \chi_3, \chi_4, \chi_5, \chi_{13}, \chi_{17} \in \text{Irr}_{\mathbb{C}}(U), \\ \chi|_{Y_1} &= 2\chi_3 + \chi_6 + \chi_8 + \chi_9 + \chi_{19}, \text{ where } \chi_3, \chi_6, \chi_8, \chi_9, \chi_{19} \in \text{Irr}_{\mathbb{C}}(Y_1), \\ \chi|_{Y_2} &= \chi_2 + \chi_6 + \chi_{15}, \text{ where } \chi_2, \chi_6, \chi_{15} \in \text{Irr}_{\mathbb{C}}(Y_2), \\ \chi|_{Y_3} &= \chi_5 + \chi_6 + \chi_{24}, \text{ where } \chi_5, \chi_6, \chi_{24} \in \text{Irr}_{\mathbb{C}}(Y_3), \\ \chi|_{Y_4} &= \chi_1 + \chi_2 + 3\chi_7, \text{ where } \chi_1, \chi_2, \chi_7 \in \text{Irr}_{\mathbb{C}}(Y_4). \end{aligned}$$

Using now the orders of the centralizers of all representatives of the conjugacy classes of G , which are also given in Proposition 3.11, one can calculate the inner product $(\chi, \chi)_G = 1$. Hence, χ is an irreducible character of G with degree $\chi(1) = 22$ by Corollary 8.12 of Isaacs [13, p. 131].

By Lemma 4.1, there is only one compatible pair $(\eta, \tau) \in \text{mfchar}(H) \times \text{mfchar}(E)$, where $E = N_G(A)$. Since $G = \langle H, E \rangle$ by Proposition 3.7(b), it follows from Thompson’s Theorem [22, p. 148] that G has only one character of degree 22. \square

Theorem 5.2. *Let H be the finite group of even order defined in Lemma 2.1. Then each finite simple group G of HS-type is isomorphic to the simple group*

$$\mathfrak{G} = \langle \mathcal{R}, \mathcal{S}, \mathcal{C}, \mathcal{G} \rangle \leq \text{GL}_{22}(11) \text{ of order } |\mathfrak{G}| = 2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$$

constructed in Theorem 4.2.

Proof. By Theorem 4.2, the group $\mathfrak{G} = \langle \mathcal{R}, \mathcal{S}, \mathcal{C}, \mathcal{G} \rangle \leq \text{GL}_{22}(11)$ is a simple group of HS-type having a 2-central involution $\mathcal{Z} = S^4$ satisfying $C_{\mathfrak{G}}(\mathcal{Z}) \cong H$. Furthermore, the group \mathfrak{G} satisfies conditions (1)–(5) in Theorem 2.1 of [18] for the prime $p = 2$ by Lemma 2.1(n), (d), Proposition 3.7(b) and Lemma 4.1, respectively.

Let G be any finite simple group of HS-type. Then G has a unique conjugacy class of 2-central involutions z by Proposition 3.4. Now Proposition 3.7(a) states that $|G| = |\mathfrak{G}| = 2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$. By Proposition 5.1, G has a faithful irreducible FG -module W of $\dim_F(W) = 22$ over the prime splitting field $F = GF(11)$ of characteristic 11. Let $A = \langle z, t, u \rangle$ be the unique maximal elementary abelian

\mathcal{CG} has order 11 by Theorem 4.2(a) and belongs to the conjugacy class 11_A of \mathfrak{G} . Using MAGMA again, it follows that $(\mathcal{CG})^\Gamma \in 11_B$. Thus, Γ induces an outer automorphism of \mathfrak{G} .

(b) By Theorem 4.2(a), the three conjugacy classes $5_A, 5_B$ and 5_C of \mathfrak{G} have centralizers with three different orders. Hence, they cannot be fused by any automorphism α of \mathfrak{G} . As $\mathcal{C} \in 5_A$, it follows that $\mathcal{C}^\alpha = \mathcal{C}^x$ for some $x \in \mathfrak{G}$. So we may assume that $\alpha \in C_{\widehat{\mathfrak{G}}}(\mathcal{C})$, where $\widehat{\mathfrak{G}}$ denotes the automorphism group of \mathfrak{G} . As \mathcal{G} has order 3 and \mathfrak{G} has only one conjugacy class of elements of order 3 by Theorem 4.2(a), it follows that $\mathcal{G}^\alpha \in \mathcal{G}^\mathfrak{G}$.

Using MAGMA, it can be checked that the elements $\mathcal{GC}, \mathcal{GCG}$ and $(\mathcal{C}^2\mathcal{G})^2\mathcal{CGC}^2$ have orders 11, 15 and 7, respectively. Furthermore,

$$\left| \left\{ \mathcal{X} \in \mathcal{G}^\mathfrak{G} \mid (\mathcal{X}\mathcal{C})^{11} = 1, (\mathcal{X}\mathcal{C}\mathcal{X})^{15} = 1, [(\mathcal{C}^2\mathcal{X})^2\mathcal{C}\mathcal{X}\mathcal{C}^2]^7 = 1 \right\} \right| = 1000,$$

and $|C_{\mathfrak{G}}(\mathcal{C}) : (C_{\mathfrak{G}}(\mathcal{G}) \cap C_{\mathfrak{G}}(\mathcal{C}))| = 500$ by MAGMA. Hence, $|\text{Out}(\mathfrak{G})| \leq 2$, and (b) holds by (a).

(c) It follows immediately from Theorem 5.2 and (b). □

Theorem 6.2. *Let X be a group generated by $z, v_1, v_2, v_3, v_4, v_5, v_6, h, k, g$ satisfying the following set $\mathcal{R}(X)$ of defining relations:*

$$\begin{aligned} z^2 = v_1^2 = v_2^2 = v_3^2 = v_4^2 = v_5^2 = v_6^2 = h^5 = k^2 = 1, \\ [z, v_1] = [z, v_2] = [z, v_3] = [z, v_4] = [z, v_5] = [z, v_6] = 1, \\ [v_1, v_2] = [v_1, v_3] = [v_1, v_4] = [v_1, v_6] = [v_1, v_5]z = 1, \\ [v_2, v_3] = [v_2, v_4] = [v_2, v_5] = [v_2, v_6]z = [v_3, v_5] = [v_3, v_4]z = [v_3, v_6]z = 1, \\ [v_4, v_5] = [v_4, v_6] = [v_5, v_6] = [z, h] = [z, k] = 1, \\ v_1^h v_2 v_4 v_5 z = v_2^h v_2 v_3 v_6 = v_3^h v_1 v_3 v_4 v_5 = 1, \\ v_4^h v_2 v_4 z = v_5^h v_1 v_2 v_4 = v_6^h v_4 = 1, \quad [v_1, k] = [v_3, k] = v_2^k v_1 v_2 = [v_4, k] = 1, \\ v_5^k v_4 v_5 v_6 z = [v_6, k] = [hk]^4 = [h, k]^3 = [k, h^2]^2 = 1, \\ g^3 = 1, \quad z^g = v_1, \quad v_1^g = zv_1, \quad v_2^g = v_2, \quad v_3^g = v_3, \\ g^{-1}v_6h^{-1}(v_2gh)^2 = xzv_4v_6v_2gh^{-1}v_2g^{-1}hv_2g^{-1}kv_6v_2v_6hkh^{-1}v_5k, \\ ghv_6 = xh^{-1}kh^{-1}v_1h^{-1}kv_2g^{-1}h, \quad \text{where } x = v_2h^3zv_3kh^4zv_3khzv_5. \end{aligned}$$

Then $X \cong \text{Aut}(\text{HS})$.

Proof. By Theorem 6.1, $\text{Aut}(\text{HS}) \cong \widehat{\mathfrak{G}} = \langle \mathcal{R}, \mathcal{S}, \mathcal{C}, \mathcal{G}, \Gamma \rangle \leq \text{GL}_{22}(11)$. In $\widehat{\mathfrak{G}}$, let

$$\begin{aligned} \varphi(z) &= \mathcal{S}^4, & \varphi(v_5) &= \mathcal{C} \cdot \mathcal{R}^{-1} \cdot \mathcal{S}^{-1} \cdot \mathcal{C} \cdot \mathcal{S}^{-1} \cdot \Gamma, \\ \varphi(v_1) &= \mathcal{S} \cdot \Gamma \cdot \mathcal{S}^{-1} \cdot \Gamma, & \varphi(v_6) &= \mathcal{R} \cdot \Gamma \cdot \mathcal{S}^{-1} \cdot \mathcal{C} \cdot \mathcal{R} \cdot \mathcal{S}^{-1} \cdot \mathcal{C}, \\ \varphi(v_2) &= \Gamma \cdot \mathcal{S}^{-1} \cdot \Gamma \cdot \mathcal{S}, & \varphi(h) &= \mathcal{R} \cdot \mathcal{C}^{-1} \cdot \mathcal{R} \cdot \mathcal{S} \cdot \mathcal{C} \cdot \mathcal{S} \cdot \mathcal{C}, \\ \varphi(v_3) &= \mathcal{S}^2 \cdot \Gamma \cdot \mathcal{S}^{-2}, & \varphi(k) &= \Gamma \cdot \mathcal{C} \cdot \mathcal{R}^{-1} \cdot \mathcal{S}^{-1} \cdot \mathcal{C} \cdot \mathcal{R}, \\ \varphi(v_4) &= \mathcal{R} \cdot \mathcal{C} \cdot \mathcal{S}^{-1} \cdot \mathcal{C} \cdot \mathcal{S} \cdot \Gamma, & \varphi(q) &= \mathcal{S} \cdot \mathcal{R} \cdot \mathcal{S} \cdot \mathcal{R}^{-1} \cdot \mathcal{G} \cdot \mathcal{R}^{-1}. \end{aligned}$$

Then it has been checked by means of MAGMA that $\varphi : X \rightarrow \widehat{\mathfrak{G}}$ is an isomorphism. □

Appendix A. Representatives of Conjugacy Classes

A.1 Conjugacy classes of $H = C_G(z) = \langle c, r, s \rangle$

i	h_i	$ C_H(h_i) $	2P	3P	5P
1 _a	1	$2^9 \cdot 3 \cdot 5$	1 _a	1 _a	1 _a
2 _a	$(s)^4$	$2^9 \cdot 3 \cdot 5$	1 _a	2 _a	2 _a
2 _b	$(rs)^2$	2^8	1 _a	2 _b	2 _b
2 _c	$(rsc)^3$	$2^5 \cdot 3$	1 _a	2 _c	2 _c
2 _d	$(rcs)^3$	$2^5 \cdot 3$	1 _a	2 _d	2 _d
2 _e	r^2cs^2	2^6	1 _a	2 _e	2 _e
3 _a	r	$2^3 \cdot 3$	3 _a	1 _a	3 _a
4 _a	$(rc)^5$	$2^8 \cdot 3 \cdot 5$	2 _a	4 _a	4 _a
4 _b	$(rc^2)^2$	2^8	2 _a	4 _b	4 _b
4 _c	$(sc)^2$	2^7	2 _b	4 _c	4 _c
4 _d	r^2src^2s	2^7	2 _b	4 _d	4 _d
4 _e	$(s)^2$	2^6	2 _a	4 _e	4 _e
4 _f	r^2scs	2^6	2 _b	4 _f	4 _f
4 _g	rs	2^5	2 _b	4 _g	4 _g

i	h_i	$ C_H(h_i) $	2P	3P	5P
4 _h	sc^3	2^5	2 _b	4 _h	4 _h
5 _a	c	$2^2 \cdot 5$	5 _a	5 _a	1 _a
6 _a	$(r^2c)^2$	$2^3 \cdot 3$	3 _a	2 _a	6 _a
6 _b	rsc	$2^2 \cdot 3$	3 _a	2 _c	6 _b
6 _c	rcs	$2^2 \cdot 3$	3 _a	2 _d	6 _c
8 _a	s	2^4	4 _e	8 _a	8 _a
8 _b	sc	2^4	4 _c	8 _c	8 _b
8 _c	$(sc)^3$	2^4	4 _c	8 _b	8 _c
8 _d	rc^2	2^4	4 _b	8 _d	8 _d
8 _e	rs^2cs	2^4	4 _e	8 _e	8 _e
10 _a	$(rc)^2$	$2^2 \cdot 5$	5 _a	10 _a	2 _a
12 _a	r^2c	$2^2 \cdot 3$	6 _a	4 _a	12 _a
20 _a	rc	$2^2 \cdot 5$	10 _a	20 _a	4 _a
20 _b	$(rc)^{11}$	$2^2 \cdot 5$	10 _a	20 _b	4 _a

A.2 Conjugacy classes of $D = N_H(A) = \langle r, s \rangle$

i	d_i	$ C_D(d_i) $	2P	3P
1 _a	1	$2^9 \cdot 3$	1 _a	1 _a
2 _a	$(s)^4$	$2^9 \cdot 3$	1 _a	2 _a
2 _b	$(r^2s)^4$	2^8	1 _a	2 _b
2 _c	$(rs)^2$	2^6	1 _a	2 _c
2 _d	$r^2s^2rs^2$	2^6	1 _a	2 _d
2 _e	$r^2sr^2srs^3$	2^5	1 _a	2 _e
2 _f	$r^2sr^2s^3rs$	2^5	1 _a	2 _f
3 _a	r	$2^2 \cdot 3$	3 _a	1 _a
4 _a	$(rs^2)^3$	$2^8 \cdot 3$	2 _a	4 _a
4 _b	$(rsrs^3)^2$	2^8	2 _a	4 _b
4 _c	$(r^2s)^2$	2^7	2 _b	4 _c
4 _d	$r^2sr^2s^5$	2^7	2 _b	4 _d
4 _e	$(s)^2$	2^6	2 _a	4 _e
4 _f	r^2srs^2rs	2^6	2 _a	4 _f

i	d_i	$ C_D(d_i) $	2P	3P
4 _g	$r^2sr^2sr^2s^2$	2^6	2 _b	4 _g
4 _h	r^2srs^2	2^5	2 _b	4 _h
4 _i	r^2s^2rs	2^5	2 _b	4 _i
4 _j	rs	2^4	2 _c	4 _j
4 _k	$r^2sr^2s^2$	2^4	2 _c	4 _k
6 _a	$(rs^2)^2$	$2^2 \cdot 3$	3 _a	2 _a
8 _a	s	2^4	4 _e	8 _a
8 _b	r^2s	2^4	4 _c	8 _c
8 _c	$(r^2s)^3$	2^4	4 _c	8 _b
8 _d	$rsrs^3$	2^4	4 _b	8 _d
8 _e	r^2sr^2srs	2^4	4 _e	8 _e
12 _a	rs^2	$2^2 \cdot 3$	6 _a	4 _a
12 _b	$(rs^2)^5$	$2^2 \cdot 3$	6 _a	4 _a

A.3 Conjugacy classes of $E = N_G(A) = \langle r, s, g \rangle$

i	e_i	$ C_E(e_i) $	2P	3P	7P
1 _a	1	$2^9 \cdot 3 \cdot 7$	1 _a	1 _a	1 _a
2 _a	$(s)^4$	$2^9 \cdot 3$	1 _a	2 _a	2 _a
2 _b	$(rs)^2$	2^6	1 _a	2 _b	2 _b
2 _c	sy	2^6	1 _a	2 _c	2 _c
3 _a	r	$2^2 \cdot 3$	3 _a	1 _a	3 _a
4 _a	$(rs^2)^3$	$2^8 \cdot 3$	2 _a	4 _a	4 _a
4 _b	$(r^2s)^2$	2^8	2 _a	4 _b	4 _b
4 _c	$(s)^2$	2^6	2 _a	4 _c	4 _c
4 _d	r^2srs^2	2^6	2 _a	4 _d	4 _d
4 _e	rs	2^4	2 _b	4 _e	4 _e

i	e_i	$ C_E(e_i) $	2P	3P	7P
4 _f	ry	2^4	2 _b	4 _f	4 _f
6 _a	$(rs^2)^2$	$2^2 \cdot 3$	3 _a	2 _a	6 _a
7 _a	r^2y	7	7 _a	7 _b	1 _a
7 _b	$(r^2y)^3$	7	7 _b	7 _a	1 _a
8 _a	s	2^4	4 _c	8 _a	8 _a
8 _b	r^2s	2^4	4 _b	8 _b	8 _b
8 _c	r^2ys^2	2^4	4 _c	8 _c	8 _c
12 _a	rs^2	$2^2 \cdot 3$	6 _a	4 _a	12 _b
12 _b	$(rs^2)^5$	$2^2 \cdot 3$	6 _a	4 _a	12 _a

A.4 Conjugacy classes of $U = C_G(v) = \langle v, y_1, y_3, y_4, e \rangle$

i	u_i	$ C_U(u_i) $	2P	3P	5P
1 _a	1	$2^6 \cdot 3^2 \cdot 5$	1 _a	1 _a	1 _a
2 _a	v	$2^6 \cdot 3^2 \cdot 5$	1 _a	2 _a	2 _a
2 _b	y_3	$2^5 \cdot 3$	1 _a	2 _b	2 _b
2 _c	vy_3	$2^5 \cdot 3$	1 _a	2 _c	2 _c
2 _d	y_4	$2^4 \cdot 5$	1 _a	2 _d	2 _d
2 _e	vy_4	$2^4 \cdot 5$	1 _a	2 _e	2 _e
2 _f	y_1	2^5	1 _a	2 _f	2 _f
2 _g	vy_1	2^6	1 _a	2 _g	2 _g
3 _a	e	$2^2 \cdot 3^2$	3 _a	1 _a	3 _a
4 _a	y_1y_3	2^5	2 _f	4 _a	4 _a
4 _b	$(y_1y_4)^2$	2^5	2 _f	4 _b	4 _b
4 _c	vy_1y_3	2^5	2 _f	4 _c	4 _c
4 _d	$vy_1y_4y_1y_4$	2^5	2 _f	4 _d	4 _d

i	u_i	$ C_U(u_i) $	2P	3P	5P
4 _e	$y_1y_3y_4$	2^4	2 _f	4 _e	4 _e
4 _f	$vy_1y_3y_4$	2^4	2 _f	4 _f	4 _f
5 _a	$(y_4e)^2$	$2^2 \cdot 5$	5 _a	5 _a	1 _a
6 _a	ve	$2^2 \cdot 3^2$	3 _a	2 _a	6 _a
6 _b	$y_3y_4ey_4$	$2^2 \cdot 3$	3 _a	2 _b	6 _b
6 _c	$vy_3y_4ey_4$	$2^2 \cdot 3$	3 _a	2 _c	6 _c
8 _a	y_1y_4	2^4	4 _b	8 _a	8 _a
8 _b	y_3y_4	2^4	4 _b	8 _b	8 _b
8 _c	vy_1y_4	2^4	4 _b	8 _c	8 _c
8 _d	vy_3y_4	2^4	4 _b	8 _d	8 _d
10 _a	y_4e	$2^2 \cdot 5$	5 _a	10 _a	2 _d
10 _b	vy_4e	$2^2 \cdot 5$	5 _a	10 _b	2 _e
10 _c	vy_4ey_4e	$2^2 \cdot 5$	5 _a	10 _c	2 _a

A.5 Conjugacy classes of $N_0 = N_G(3_A) = \langle a, e, y_3, k \rangle$

i	x_i	$ C_{N_0}(x_i) $	2P	3P	5P
1 _a	1	$2^4 \cdot 3^2 \cdot 5$	1 _a	1 _a	1 _a
2 _a	y_3	$2^4 \cdot 3 \cdot 5$	1 _a	2 _a	2 _a
2 _b	$(ka)^3$	$2^3 \cdot 3^2$	1 _a	2 _b	2 _b
2 _c	$(a)^2$	$2^4 \cdot 3$	1 _a	2 _c	2 _c
2 _d	$(y_3ka)^3$	$2^3 \cdot 3$	1 _a	2 _d	2 _d
2 _e	y_3a^2	2^4	1 _a	2 _e	2 _e
3 _a	e	$2^3 \cdot 3^2 \cdot 5$	3 _a	1 _a	3 _a
3 _b	k	$2^2 \cdot 3^2$	3 _b	1 _a	3 _b
3 _c	ek	$2 \cdot 3^2$	3 _c	1 _a	3 _c
4 _a	a	$2^3 \cdot 3$	2 _c	4 _a	4 _a
4 _b	y_3a	2^3	2 _c	4 _b	4 _b

i	x_i	$ C_{N_0}(x_i) $	2P	3P	5P
5 _a	$kaka^3$	$2 \cdot 3 \cdot 5$	5 _a	5 _a	1 _a
6 _a	ka	$2^2 \cdot 3^2$	3 _b	2 _b	6 _a
6 _b	$ekakaka$	$2^2 \cdot 3^2$	3 _a	2 _b	6 _b
6 _c	$(ea)^2$	$2^3 \cdot 3$	3 _a	2 _c	6 _c
6 _d	eka	$2 \cdot 3^2$	3 _c	2 _b	6 _d
6 _e	y_3k	$2^2 \cdot 3$	3 _b	2 _a	6 _e
6 _f	y_3ka	$2^2 \cdot 3$	3 _b	2 _d	6 _f
10 _a	y_3kaka^3	$2 \cdot 5$	5 _a	10 _a	2 _a
12 _a	ea	$2^2 \cdot 3$	6 _c	4 _a	12 _a
15 _a	$ekaka^3$	$3 \cdot 5$	15 _a	5 _a	3 _a

A.6 Conjugacy classes of $N_1 = N_G(5_A) = \langle a_1, b_1, x, y \rangle$

i	n_i	$ C_{N_1}(n_i) $	2P	5P
1 _a	1	$2^4 \cdot 5^3$	1 _a	1 _a
2 _a	$(a_1)^4$	$2^4 \cdot 5$	1 _a	2 _a
2 _b	b_1	$2^3 \cdot 5$	1 _a	2 _b
4 _a	$a_1^2b_1$	$2^3 \cdot 5$	2 _a	4 _a
4 _b	$(a_1)^2$	2^4	2 _a	4 _b
4 _c	$(a_1)^6$	2^4	2 _a	4 _c
5 _a	$(a_1^2b_1xy)^4$	$2^2 \cdot 5^3$	5 _a	1 _a
5 _b	x	$2 \cdot 5^2$	5 _b	1 _a
5 _c	xy	5^2	5 _c	1 _a

i	n_i	$ C_{N_1}(n_i) $	2P	5P
8 _a	a_1	2^3	4 _b	8 _a
8 _b	$(a_1)^3$	2^3	4 _c	8 _b
8 _c	a_1b_1	2^3	4 _c	8 _c
8 _d	$(a_1b_1)^3$	2^3	4 _b	8 _d
10 _a	$(a_1^2b_1xy)^2$	$2^2 \cdot 5$	5 _a	2 _a
10 _b	b_1y	$2 \cdot 5$	5 _b	2 _b
20 _a	$a_1^2b_1xy$	$2^2 \cdot 5$	10 _a	4 _a
20 _b	$(a_1^2b_1xy)^{11}$	$2^2 \cdot 5$	10 _a	4 _a

A.7 Conjugacy classes of $N_2 = N_G(5_B) = \langle c_2, y_3, w, v \rangle$

i	m_i	$ C_{N_2}(m_i) $	2P	3P	5P
1 _a	1	$2^4 \cdot 3 \cdot 5^2$	1 _a	1 _a	1 _a
2 _a	$(y_3)^2$	$2^4 \cdot 3 \cdot 5$	1 _a	2 _a	2 _a
2 _b	v	$2^4 \cdot 5$	1 _a	2 _b	2 _b
2 _c	$y_3^2 v$	2^4	1 _a	2 _c	2 _c
3 _a	wv	$2^2 \cdot 3 \cdot 5$	3 _a	1 _a	3 _a
4 _a	y_3	$2^4 \cdot 3 \cdot 5$	2 _a	4 _b	4 _a
4 _b	$(y_3)^3$	$2^4 \cdot 3 \cdot 5$	2 _a	4 _a	4 _b
4 _c	$y_3 v$	2^4	2 _a	4 _d	4 _c
4 _d	$(y_3 v)^3$	2^4	2 _a	4 _c	4 _d
5 _a	c_2	$2^2 \cdot 3 \cdot 5^2$	5 _a	5 _a	1 _a
5 _b	w	$2^2 \cdot 5^2$	5 _c	5 _c	1 _a
5 _c	$(w)^2$	$2^2 \cdot 5^2$	5 _b	5 _b	1 _a
5 _d	$c_2 w$	5^2	5 _e	5 _e	1 _a

i	m_i	$ C_{N_2}(m_i) $	2P	3P	5P
5 _e	$(c_2 w)^2$	5^2	5 _d	5 _d	1 _a
6 _a	$(y_3 w v)^2$	$2^2 \cdot 3$	3 _a	2 _a	6 _a
10 _a	$c_2 v$	$2^2 \cdot 5$	5 _a	10 _a	2 _b
10 _b	$(y_3 w)^2$	$2^2 \cdot 5$	5 _b	10 _c	2 _a
10 _c	$(y_3 w)^6$	$2^2 \cdot 5$	5 _c	10 _b	2 _a
12 _a	$y_3 w v$	$2^2 \cdot 3$	6 _a	4 _b	12 _a
12 _b	$(y_3 w v)^7$	$2^2 \cdot 3$	6 _a	4 _a	12 _b
15 _a	$c_2 w v$	$3 \cdot 5$	15 _a	5 _a	3 _a
20 _a	$y_3 v$	$2^2 \cdot 5$	10 _b	20 _b	4 _a
20 _b	$(y_3 w)^3$	$2^2 \cdot 5$	10 _c	20 _a	4 _b
20 _c	$(y_3 w)^{11}$	$2^2 \cdot 5$	10 _b	20 _d	4 _b
20 _d	$(y_3 w)^{13}$	$2^2 \cdot 5$	10 _c	20 _c	4 _a

Appendix B. Character Tables

B.1 Character table of $H = C_G(z)$

2	9	9	8	5	5	6	3	8	8	7	7	6	6	5	5	2	3	2	2	4	4	4	4	4	2	2	2	2
3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

1a	2a	2b	2c	2d	2e	3a	4a	4b	4c	4d	4e	4f	4g	4h	5a	6a	6b	6c	8a	8b	8c	8d	8e	10a	12a	20a	20b
2P 1a	1a	1a	1a	1a	1a	3a	2a	2a	2b	2b	2b	2a	2b	2b	5a	3a	3a	3a	4d	4f	4d	4b	4f	5a	6a	10a	10a
3P 1a	2a	2b	2c	2d	2e	1a	4a	4b	4c	4d	4e	4f	4g	4h	5a	2a	2c	2d	8c	8b	8a	8d	8e	10a	4a	20a	20b
5P 1a	2a	2b	2c	2d	2e	3a	4a	4b	4c	4d	4e	4f	4g	4h	1a	6a	6b	6c	8a	8b	8c	8d	8e	2a	12a	4a	4a

X.1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.2	1	1	1	-1	-1	1	1	1	1	1	1	1	1	-1	-1	1	1	-1	-1	-1	-1	-1	-1	1	1	1	1	1
X.3	4	4	4	2	2	1	4	4	4	4	4	4	4	2	2	-1	1	-1	-1	1	1	1	1	1	1	1	1	1
X.4	4	4	4	-2	-2	1	4	4	4	4	4	4	4	-2	-2	-1	1	1	1	1	1	1	1	1	1	1	1	1
X.5	5	5	5	1	1	1	-1	5	5	5	5	5	5	1	1	1	1	1	1	-1	1	1	-1	-1	1	1	1	1
X.6	5	5	5	-1	-1	1	-1	5	5	5	5	5	5	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
X.7	6	6	6	6	6	-2	6	6	6	-2	-2	-2	-2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.8	6	6	-2	6	-2	6	-6	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.9	6	6	-2	6	-2	6	-6	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.10	6	6	-2	6	-2	6	-6	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.11	6	6	-2	6	-2	6	-6	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.12	8	-8	8	8	8	2	8	8	8	8	8	8	8	-2	-2	1	1	1	1	1	1	1	1	1	1	1	1	1
X.13	10	10	2	2	-2	2	1	-10	-2	2	2	2	-2	-2	1	-1	1	1	1	1	1	1	1	1	1	1	1	1
X.14	10	10	2	-2	2	2	1	-10	-2	2	2	2	-2	-2	1	1	-1	1	1	1	1	1	1	1	1	1	1	1
X.15	10	10	2	4	-4	-2	1	-10	-2	-2	-2	-2	2	2	1	1	-1	1	1	1	1	1	1	1	1	1	1	1
X.16	10	10	2	-4	4	-2	1	-10	-2	-2	-2	-2	2	2	1	-1	1	1	1	1	1	1	1	1	1	1	1	1
X.17	15	15	-1	3	3	-1	15	-1	3	3	3	3	-1	-1	1	-1	1	1	1	1	1	1	1	1	1	1	1	1
X.18	15	15	-1	-3	-3	-1	15	-1	3	3	3	3	-1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.19	15	15	-1	3	3	3	15	-1	-1	-1	-1	-1	3	-1	-1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.20	15	15	-1	-3	-3	3	15	-1	-1	-1	-1	-1	3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.21	20	20	4	-2	2	-1	-20	-4	2	2	2	2	2	-2	-2	-1	1	-1	1	1	1	1	1	1	1	1	1	1
X.22	20	20	4	2	-2	-1	-20	-4	2	2	2	2	2	-2	-2	-1	-1	1	1	1	1	1	1	1	1	1	1	1
X.23	24	-24	24	24	24	4	-4	-4	-4	-4	-4	-4	-4	-1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.24	24	-24	24	24	24	4	-4	-4	-4	-4	-4	-4	-4	-1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.25	24	24	-8	24	24	-24	8	8	8	8	8	8	8	-1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.26	30	30	-2	30	30	-2	30	-2	-2	-2	-2	-2	-2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
X.27	32	-32	32	32	32	2	-2	-2	-2	-2	-2	-2	-2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
X.28	40	-40	40	40	40	2	-4	4	4	4	4	4	4	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2

$A = 2i$ and $B = -i\sqrt{5}$.

B.2 Character table of $D = N_H(A)$

	2	9	9	8	6	6	5	5	2	8	8	7	7	6	6	6	5	5	4	4	2	4	4	4	4	4	2	2
	3	1	1	1	1	1	1	1
	1a	2a	2b	2c	2d	2e	2f	3a	4a	4b	4c	4d	4e	4f	4g	4h	4i	4j	4k	6a	8a	8b	8c	8d	8e	12a	12b	
2P	1a	1a	1a	1a	1a	1a	1a	3a	2a	2a	2b	2b	2a	2a	2b	2b	2d	2d	3a	4f	4b	4d	4f	4d	6a	6a		
3P	1a	2a	2b	2c	2d	2e	2f	1a	4a	4b	4c	4d	4e	4f	4g	4h	4i	4j	4k	2a	8a	8b	8c	8d	8c	4a	4a	
X.1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
X.2	1	1	1	1	-1	-1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	1	-1	1	-1	-1	-1	1	1		
X.3	2	2	2	2	.	.	-1	2	2	2	2	2	2	-1	.	2	-1	-1		
X.4	3	3	3	-1	3	-1	-1	.	3	3	-1	-1	-1	3	-1	-1	-1	.	1	-1	1	1	1	.	.	.		
X.5	3	3	3	-1	3	1	1	.	3	3	-1	-1	-1	3	1	1	1	1	.	-1	-1	-1	-1	.	.	.		
X.6	3	3	3	-1	-1	-1	-1	.	3	3	3	3	-1	-1	-1	1	1	.	1	-1	-1	1	-1	.	.	.		
X.7	3	3	3	-1	1	1	1	.	3	3	3	3	-1	-1	1	1	-1	.	-1	-1	1	-1	1	.	.	.		
X.8	3	3	3	3	-1	-1	-1	.	3	3	-1	-1	-1	3	-1	-1	1	1	.	-1	-1	1	-1	.	.	.		
X.9	3	3	3	3	-1	1	1	.	3	3	-1	-1	-1	3	-1	1	1	-1	.	1	-1	1	-1	.	.	.		
X.10	4	4	4	.	.	-2	2	1	-4	-4	-2	2	.	1	-1	-1		
X.11	4	4	4	.	.	-2	2	1	-4	-4	2	-2	.	1	-1	-1		
X.12	6	6	-2	2	-2	.	.	.	-6	2	-2	-2	2	2	A	-A	.		
X.13	6	6	-2	2	-2	.	.	.	-6	2	-2	-2	2	2	-A	A	.		
X.14	6	6	-2	2	-2	2	.	.	-6	2	-2	-2	2	2	-2	-2		
X.15	6	6	-2	2	2	-2	.	.	-6	2	-2	-2	2	2	2	-2		
X.16	6	6	-2	2	-2	.	.	.	-6	2	2	-2	2	2	-2	.	2	.		
X.17	6	6	-2	2	-2	.	.	.	-6	2	2	-2	2	2	2	.	-2	.		
X.18	6	6	-2	2	2	.	.	.	-6	2	2	-2	-2	-2	.	.	.	-2	2		
X.19	6	6	-2	2	2	.	.	.	-6	2	2	-2	-2	-2	.	.	.	2	-2		
X.20	6	6	6	-2	-2	.	.	.	6	6	-2	-2	-2	-2	2	.	.	.		
X.21	8	8	8	.	.	.	-1	-8	-8	-1	1	1		
X.22	8	-8	2	.	-4	4	-2		
X.23	8	-8	-1	.	-4	4	1	B	-B		
X.24	8	-8	-1	.	-4	4	1	-B	B		
X.25	12	12	-4	.	.	2	2	.	12	-4	-2	-2		
X.26	12	12	-4	.	.	-2	-2	.	12	-4	2	2		
X.27	24	-24	4	-4		

$A = 2i$ and $B = -\sqrt{3}$.

B.3 Character table of $E = N_G(A)$

	2	9	9	6	6	2	8	8	6	6	4	4	2	.	.	4	4	4	2	2
	3	1	1	.	.	1	1	1	1	1
	7	1	1	1	
	1a	2a	2b	2c	3a	4a	4b	4c	4d	4e	4f	6a	7a	7b	8a	8b	8c	12a	12b	
2P	1a	1a	1a	1a	3a	2a	2a	2a	2a	2c	2c	3a	7a	7b	4b	4d	4d	6a	6a	
3P	1a	2a	2b	2c	1a	4a	4b	4c	4d	4e	4f	2a	7b	7a	8a	8b	8c	4a	4a	
7P	1a	2a	2b	2c	3a	4a	4b	4c	4d	4e	4f	6a	1a	8a	8b	8c	12b	12a		
X.1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
X.2	3	3	-1	-1	.	3	3	-1	-1	1	1	.	A	/A	-1	1	1	.	.	
X.3	3	3	-1	-1	.	3	3	-1	-1	1	1	.	/A	A	-1	1	1	.	.	
X.4	6	6	2	2	.	6	6	2	2	.	.	.	-1	1	2	
X.5	7	7	3	-1	1	-1	-1	-1	3	-1	-1	1	.	.	-1	1	1	-1	-1	
X.6	7	7	-1	3	1	-1	-1	3	-1	1	1	1	.	.	-1	-1	-1	-1	-1	
X.7	7	7	-1	-1	1	7	7	-1	-1	-1	-1	1	.	.	-1	-1	-1	1	1	
X.8	8	8	.	.	-1	8	8	-1	1	1	.	.	.	-1	-1	
X.9	14	14	2	2	-1	-2	-2	2	2	.	.	-1	.	.	-2	.	.	1	1	
X.10	14	-2	-2	2	2	-6	2	-2	2	.	.	-2	
X.11	14	-2	-2	2	-1	-6	2	-2	2	.	.	1	B	-B	
X.12	14	-2	-2	2	-1	-6	2	-2	2	.	.	1	-B	B	
X.13	21	21	1	-3	.	-3	-3	-3	1	1	1	.	.	.	1	-1	-1	.	.	
X.14	21	21	-3	1	.	-3	-3	1	-3	-1	-1	.	.	.	1	1	1	.	.	
X.15	42	-6	-2	-2	.	6	-2	2	2	-2	2	.	
X.16	42	-6	-2	-2	.	6	-2	2	2	2	-2	.	.	
X.17	42	-6	2	2	.	6	-2	-2	-2	2	2	
X.18	42	-6	2	2	.	6	-2	-2	-2	2	2	
X.19	42	-6	2	-2	.	-18	6	2	-2	

$A = \frac{1}{2}(-1 + i\sqrt{7})$ and $B = \sqrt{3}$.

B.4 Character table of $C_G(2B)$

	2	6	6	5	5	4	4	6	6	2	5	5	5	5	4	4	2	2	2	2	4	4	4	4	2	2	2			
3P	2	2	1	1	1	1	1	1	1	2	1	1	1	1	1	1	2	1	1	1	1	1	1	1	1	1	1	1		
5P	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1		
	1a	2a	2b	2c	2d	2e	2f	2g	3a	4a	4b	4c	4d	4e	4f	5a	6a	6b	6c	8a	8b	8c	8d	10a	10b	10c				
2P	1a	1a	1a	1a	1a	1a	1a	1a	3a	2g	2g	2g	2g	2g	5a	3a	3a	3a	3a	4d	4d	4d	4d	5a	5a	5a				
3P	1a	2a	2b	2c	2d	2e	2f	2g	1a	4a	4b	4c	4d	4e	4f	5a	2a	2b	2c	8a	8b	8c	8d	10a	10b	10c				
5P	1a	2a	2b	2c	2d	2e	2f	2g	3a	4a	4b	4c	4d	4e	4f	1a	6a	6b	6c	8a	8b	8c	8d	2d	2a	2e				
X.1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1		
X.2	1	1	1	1	-1	-1	1	1	1	1	1	1	1	1	-1	-1	1	1	1	1	-1	-1	-1	-1	-1	1	-1	1	-1	
X.3	1	-1	-1	1	-1	1	-1	1	1	-1	-1	1	1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	
X.4	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	-1	
X.5	1	1	-1	-1	-1	1	1	1	1	-1	1	-1	1	1	1	1	-1	-1	-1	1	1	-1	-1	1	1	-1	1	1	-1	
X.6	1	1	-1	-1	1	1	1	1	1	-1	1	-1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	1	1	1	1	
X.7	1	-1	1	-1	1	-1	1	1	1	1	-1	-1	1	-1	1	-1	1	-1	-1	1	-1	-1	1	-1	1	-1	1	1	-1	
X.8	1	-1	1	-1	1	-1	1	1	1	-1	-1	1	1	-1	1	-1	1	-1	-1	1	-1	-1	1	-1	1	-1	1	1	-1	
X.9	9	-9	-3	3	-1	1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	-1	-1	1	1	1	-1	1	1	1	
X.10	9	-9	-3	3	1	-1	-1	1	1	1	-1	-1	1	-1	1	1	-1	-1	1	-1	-1	1	1	1	1	-1	1	1	-1	
X.11	9	-9	3	-3	-1	1	-1	1	1	-1	1	1	-1	-1	1	1	-1	-1	1	-1	-1	1	1	1	-1	1	1	1	1	
X.12	9	-9	3	-3	1	-1	-1	1	1	-1	1	1	-1	-1	1	1	-1	-1	1	-1	-1	1	1	1	-1	1	1	1	-1	
X.13	9	9	3	3	-1	-1	1	1	1	-1	1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1
X.14	9	9	3	3	1	1	1	1	1	-1	-1	-1	-1	-1	1	1	-1	-1	1	1	1	1	1	1	1	1	1	1	-1	-1
X.15	9	9	-3	-3	-1	-1	1	1	1	1	1	1	-1	-1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	-1	-1
X.16	9	9	-3	-3	1	1	1	1	1	1	1	1	1	1	-1	1	1	-1	-1	-1	-1	-1	-1	-1	1	-1	1	1	-1	
X.17	10	-10	-2	2	2	2	2	2	2	-2	-2	-2	-2	-2	1	-1	1	-1	1	1	1	1	1	1	1	1	1	1	1	1
X.18	10	-10	2	-2	2	2	2	2	2	2	2	-2	-2	-2	1	-1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.19	10	10	2	2	2	2	2	2	2	2	-2	-2	-2	-2	1	-1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.20	10	10	-2	-2	2	2	2	2	2	-2	-2	-2	-2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.21	16	16	4	4	4	4	4	4	4	-2	-2	-2	-2	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1
X.22	16	16	-4	-4	-4	-4	-4	-4	-4	-2	-2	-2	-2	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1
X.23	16	-16	-4	4	4	4	4	4	4	-2	-2	-2	-2	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
X.24	16	-16	4	-4	-4	-4	-4	-4	-4	-2	-2	-2	-2	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
X.25	20	20	4	4	4	4	4	4	4	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
X.26	20	-20	4	-4	-4	-4	-4	-4	-4	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2

B.5 Character table of $N_G(3A)$

	2	4	4	3	4	3	4	3	2	1	3	3	3	1	2	2	3	1	2	2	1	2	.
3	2	1	2	1	1	1	2	2	2	1	1	1	2	2	1	2	1	2	1	1	1	1	1
5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1a	2a	2b	2c	2d	2e	3a	3b	3c	4a	4b	5a	6a	6b	6c	6d	6e	6f	10a	12a	15a		
2P	1a	1a	1a	1a	1a	1a	3a	3b	3c	2c	2c	5a	3a	3b	3c	3b	3b	5a	6c	15a			
3P	1a	2a	2b	2c	2d	2e	1a	1a	1a	4a	4b	5a	2b	2b	2c	2b	2a	2d	10a	4a	5a		
5P	1a	2a	2b	2c	2d	2e	3a	3b	3c	4a	4b	1a	6a	6b	6c	6d	6e	6f	2a	12a	3a		
X.1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
X.2	1	1	-1	-1	1	1	1	1	-1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	-1	1	
X.3	1	-1	-1	1	-1	1	1	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	-1	
X.4	1	-1	1	1	-1	-1	1	1	1	-1	1	1	1	1	1	1	1	1	-1	-1	1	1	
X.5	2	2	2	2	2	2	-1	2	-1	2	2	-1	2	-1	2	-1	2	-1	2	2	2	2	
X.6	2	-2	2	2	2	2	-1	2	-1	2	2	1	-2	-1	1	1	1	1	1	1	1	1	
X.7	4	-4	2	-2	4	4	1	1	1	1	1	-1	2	-1	-1	-1	1	1	1	1	1	1	
X.8	4	-4	-2	2	4	4	1	1	1	1	1	-1	-2	1	1	-1	-1	1	1	1	1	1	
X.9	4	4	2	2	4	4	1	1	1	1	1	-1	2	-1	-1	1	-1	-1	1	1	1	1	
X.10	4	4	-2	-2	4	4	1	1	1	1	1	-1	-2	1	1	1	1	1	1	1	1	1	
X.11	5	-5	1	1	-1	-1	5	-1	-1	-1	1	1	1	1	1	1	1	1	1	1	1	1	
X.12	5	-5	-1	-1	1	1	5	-1	-1	-1	1	1	1	1	1	1	1	1	1	1	1	1	
X.13	5	5	1	1	1	1	5	-1	-1	-1	-1	1	1	1	1	1	1	1	1	1	1	1	
X.14	5	5	-1	-1	-1	-1	5	-1	-1	-1	1	1	1	1	1	1	1	1	1	1	1	1	
X.15	6	-6	2	2	2	2	6	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	
X.16	6	6	-2	-2	-2	-2	6	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	
X.17	8	4	4	4	4	4	8	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	
X.18	8	-4	4	4	4	4	8	-4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	
X.19	10	2	2	2	2	2	10	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	
X.20	10	-2	2	2	2	2	10	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	
X.21	12	4	4	4	4	4	12	4	4	4	4	4											

$A = i$, $B = 3i$, $C = 4i$, $D = 5i$, $E = \frac{1}{2} - \frac{1}{2}\sqrt{5}$, $E^* = \frac{1}{2} + \frac{1}{2}\sqrt{5}$, $F = 2 - 2\sqrt{5}$, $F^* = 2 + 2\sqrt{5}$, $G = -\zeta^{13} - \zeta^{17}$, $H = -\zeta - \zeta^9$, where $\zeta = \exp \frac{2\pi i}{20}$.

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