Journal of Algebra and Its Applications
Vol. 6, No. 1 (2007) 135–171

O'NAN GROUP UNIQUELY DETERMINED BY THE CENTRALIZER OF A 2-CENTRAL INVOLUTION

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Received 1 August 2005
Accepted 8 March 2006

Communicated by D. Passman

In this paper we give a self-contained existence and uniqueness proof for the sporadic O'Nan group ON by showing that it is uniquely determined up to isomorphism by the centralizer H of a 2-central involution z. We establish for such a simple group G a presentation in terms of generators and defining relations and a faithful permutation representation of degree 2.624.832 with a uniquely determined stabilizer isomorphic to the small sporadic Janko group J_1. We also calculate its character table by new methods and determine a system of representatives of the conjugacy classes of G.

1. Introduction

The Brauer–Fowler Theorem asserts that there are only finitely many non-isomorphic finite simple groups G which have 2-central involution z such that $C_G(z) \cong H$, where H is a fixed group with a center $Z(H)$ of even order. In [11] a simple group G is said to be uniquely determined if it is (up to isomorphism) the only simple group Y having a 2-central involution y such that $C_Y(y) \cong C_G(z) \cong H$. Suzuki's survey article [16] shows that many simple groups are uniquely determined. However, that paper also mentions several examples of simple groups which are not uniquely determined, e.g. the centralizers of the 2-central involutions of the 3 non-isomorphic simple groups $L_5(2)$, M_{24} and Held's sporadic simple group He are isomorphic.

In this article we show that the sporadic simple group ON is uniquely determined, see Theorem 5.2. We also give an existence proof, calculate its character table and provide a classification of all its conjugacy classes in terms of short words of the 4 generators of a new presentation of ON, see Theorems 5.2 and 7.4 and Corollary 6.4, respectively.

The simple sporadic group $G = \mathsf{ON}$ was discovered by O'Nan in his highly original paper [13], where he established the character table and the group structure of the normalizers $N_G(p)$ of elements p of prime order dividing the group order $|G| = 2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$. In particular, he showed that such a group $G = \mathsf{ON}$ has only one conjugacy class z^G of involutions. Furthermore, $H = C_G(z)$ is of shape $4L_3(4) : 2$ by his Lemma 4.8 of [13]. All these properties of ON were derived by him from a presentation of a given Sylow 2-subgroup of ON. But he himself neither proved the existence nor the uniqueness of ON.

Using the deep results of [13] Soicher gave in [15] a short computer aided, but checkable, proof of the existence of the simple O'Nan group ON as a finitely presented group with a beautiful set of defining relations. Furthermore, he showed that a simple group G is isomorphic to ON if and only if a Sylow 2-subgroup of G is isomorphic to the one of ON. This (stronger) uniqueness theorem is an immediate consequence of Theorem 5.2 and Lemma 4.8 of [13], because the second author has shown by means of MAGMA there are exactly 36 non-isomorphic groups H_i of shape $4L_3(4) : 2$, and that the isomorphism type of H_i is uniquely determined by the isomorphism type of a Sylow 2-subgroup S_i of H_i.

In Lemma 2.1 we state a presentation of a centralizer $H = C_{\mathsf{ON}}(z)$ of a 2-central involution z of that example of a O'Nan group ON for which there are 2 generating permutations of degree 122760 in [18]. Throughout this article a finite simple group G is called a simple group of ON-type if it possesses a 2-central involution z such that $C_G(z) \cong H$.

This group H has center $Z(H) = \langle z \rangle$, commutator factor group $H'/Z(H') \cong L_3(4)$, and $|H : H'| = 2$. Since the simple group $L_3(4)$ has several non-isomorphic central extensions E with a cyclic center $Z(E)$ of order 4 it is necessary to describe the structure of H by generators and relations in order to avoid ambiguities. In Lemma 2.1 explicit generators of a fixed Sylow 2-subgroup S are given as short words in the generators of $H = \langle c, r, s \rangle$. Furthermore, it is shown that S has a unique maximal elementary abelian normal subgroup $A = \langle z, t, u \rangle$ with normalizer $D = N_H(A) = \langle r, s \rangle \cong 4^3 \cdot S_4$ and a unique Klein four subgroup $V = \langle z, t \rangle$. All these facts have been deduced from the presentation of H by means of a described faithful permutation representation of H having degree 448 and MAGMA [1].

In Sec. 3 we show that in any simple group G of ON-type there exists an element g of order 3 in $N_G(A)$, $g \notin H$ such that $z^g = t$, $t^g = zt$ and $u^g = u$. We prove in Proposition 3.1 that $E = N_G(A) = \langle r, s, g \rangle$ is uniquely determined by H up to isomorphism. A set $\mathcal{R}(E)$ of defining relations for E with respect to the generating set $\{r, s, g\}$ is also given.

Then we consider in any finite simple group G of ON-type the subgroup $E_1 = \langle z, t, u, r, g \rangle$ of E. By Lemma 4.1 it is isomorphic to a split extension of the elementary abelian normal subgroup $A = \langle z, t, u \rangle$ by a Frobenius group F_{21} of order 21. We also provide a set $\mathcal{R}(E_1)$ of defining relations for E_1. Furthermore, we show that there is an element f of order 5 in H such that $H_1 = \langle z \rangle \times \langle r, f, uz, tz, f \rangle$ is the unique subgroup of shape $\langle z \rangle \times A_5$ in H containing the subgroup $\langle A, r \rangle$ of H, where A_5 is the simple group of order 60.

In Proposition 4.2 we determine a set $\mathcal{R}(J)$ of defining relations for the subgroup $J = \langle H_1, E_1 \rangle = \langle r, f, g \rangle$ of any finite simple group G of ON-type. Using the Todd–Coxeter algorithm implemented in MAGMA and Janko's uniqueness theorem [8] we show that J is isomorphic to the smallest sporadic Janko group J_1.

In Sec. 5 we prove that $H = \langle r, s, c \rangle = \langle r, s, f \rangle$ and obtain a set $\mathcal{R}_1(H)$ of defining relations of H with respect to $\{r, s, f\}$. Then we show in Theorem 5.2 that

$$\mathcal{R}(G) = \mathcal{R}_1(H) \cup \mathcal{R}(E_1) \cup \mathcal{R}(J)$$

is a set of defining relations for any finite simple group G of ON-type with respect to its four generators r, s, f, and g, and that G has a faithful permutation representation of degree 2.624.832. Furthermore, we show in this theorem that the finitely presented group $G = \langle r, s, f, g \rangle$ with set $\mathcal{R}(G)$ of defining relations is simple and that $C_G(z) = \{r, s, f\} \cong H$. In particular, this result provides an existence and uniqueness proof for the O'Nan group.

Since calculations in this permutation group $\pi(G) = \langle \pi(r), \pi(s), \pi(f), \pi(g) \rangle \leq S_{2.624.832}$ are very slow, we determine in Lemma 6.1 another permutation representation of G of degree 122.760. Its stabilizer is a subgroup $L \cong L_3(7) : 2$ for which generators are given in Proposition 6.1 as words in r, s, f, and g.

Using this permutation representation we determine in Sec. 6 presentations of the normalizers $N_G(p)$ of cyclic subgroups of prime order p with $p > 3$. The case $p = 3$ is dealt with in Sec. 3. Furthermore, representatives of all conjugacy classes of G are given as short words in the 4 generators of G, see Corollary 6.2. The same has been done for all local subgroups and the Janko subgroup J of G in terms of their given generators. The classification of the conjugacy classes of these local subgroups and of the Janko subgroup J are stated in Appendix A. The fusion of their conjugacy classes into the conjugacy classes of G is given in Corollary 6.3. Their character tables are collected in Appendix B.

In [13, p. 460] O'Nan wrote: "In this section we present a character table of the group G. The calculations used to establish this table are lengthy and will not be recounted." Therefore the character table of G has been calculated in Theorem 7.4 again by means of the first author's algorithm [12], Brauer's character formula for characters of p-blocks of defect one [3] and the LLL-algorithm [10]. This calculation is short and easy to document by means of the tables stated in the appendices. In Theorem 7.4 we also describe the construction of an irreducible 495-dimensional $\mathbb{F}_3G$-module of the O'Nan group G over the prime field $\mathbb{F}_3$.

Most of the calculations were done with MAGMA [1] and GAP [14]. They are easily checkable any time on a PC because all results are formulated in terms of words in the given generators of H and the element g of G fusing the involutions z and t. To ease calculations with MAGMA convenient permutation representations have been documented in all computational arguments. Most of the algorithms implemented in MAGMA and GAP applied in this article are described in the recent handbook of computational group theory [6]. The documentation of our calculations with MAGMA can be obtained from the second author's web page `http://scienze-como.uninsubria.it/previtali/Research.html`.

Concerning notation and terminology we refer to the books by Feit [3], Gorenstein [5], Isaacs [7] and Michler [11] where all quoted theoretical results and several new algorithms are proved.

2. The Centralizers of a 2-Central Involution

In this section we give a presentation of the finite group H which is assumed to be the centralizer of a 2-central involution in a group of ON-type. We determine a Sylow 2-subgroup S of H and an elementary abelian normal subgroup A of S such that $D = N_H(A)$ is of maximal order among all normalizers $N_H(B)$ of the elementary abelian maximal subgroups B of S.

Lemma 2.1. *Let H be the finitely presented group generated by elements r, s and c subject to the following set $\mathcal{R}(H)$ of relations*:

$$r^3 = s^{16} = c^5 = (sc^{-1})^4 = (rs^2r^{-1}s^{-1})^2 = (r^{-1}s^2rs^{-1})^2 = 1,$$
$$s^{-1}r^{-1}c^{-2}s^{-1}r^{-1}crc^2 = (r^{-1}c^{-1}s^{-1}cs)^2 = rc^2rs^{-1}r^{-1}s^{-1}c^{-2}rc^{-1} = 1,$$
$$cr^{-1}s^{-3}r^{-1}c^{-1}r^{-1}scr^{-1} = sc^{-1}sr^{-1}sr^{-1}c^{-1}sc^{-1}r^{-1}c = 1,$$
$$s^5r^{-1}cr^{-1}c^{-1}sc^{-1} = sr^{-1}s^{-1}c^{-1}sr^{-1}s^{-1}rcr^{-1}c = 1,$$
$$cs^2c^{-1}sr^{-1}cr^{-1}scr = (rsr^{-1}s^{-1}r^{-1}s^{-1})^2 = (r^{-1}s^{-3}r^{-1}s^{-1})^2 = 1,$$
$$s^2r^{-1}s^{-1}r^{-1}s^{-1}r^{-1}s^3r^{-1}s = r^{-1}c^{-2}sr^{-1}s^{-1}r^{-1}sc^{-1}r^{-1}s^{-1}c = 1.$$

Let $z = (rsrs^{-1})^2, t = (rs^2)^3s^{-4}, u = (sr)^4, v_1 = (cr)^{-3}, v_2 = r^{-1}s^{-1}rs^{-1}r^{-1}s^{-2}$, $v_3 = (sr)^{-2}$. Then z, u, t are involutions, v_1, v_2, v_3 have order 4 and the following assertions hold:

(a) $\mathrm{Aut}(H) = \mathrm{Inn}(H) : K$, *where K is a Klein 4-group and there is $\gamma \in \mathrm{Aut}(H) - \mathrm{Inn}(H)$ fixing r.*
(b) $S = \langle s, s^{-1}rsr, s^2r^{-1}s^{-2}rs^{-1}\rangle$ *is a Sylow 2-subgroup of H with center $Z(S) = \langle z\rangle$ such that $\gamma(S) = S$. Moreover $s^8 = v_1^2 = z$, $v_2^2 = t$, $v_3^2 = u$, z, t, u are involutions.*
(c) $P = O_2(H) = \langle v_1\rangle$ *is cyclic of order 4.*
(d) $A = \langle z, t, u\rangle$ *is the unique elementary abelian normal subgroup of rank 3 of S.*
(e) $V = \langle z, t\rangle$ *is the unique elementary abelian normal 4-group of S.*
(f) $C_H(V) = C_S(V)$ *is a maximal subgroup of S, and $V = Z(C_S(V))$.*
(g) $S = N_H(V)$.
(h) $C = C_H(A) = C_S(A) = \langle v_1, v_2, v_3\rangle$ *is a homocyclic subgroup of rank 3 and order 64 of S.*
(i) *C is the unique abelian subgroup of its order in S.*

(j) $D = N_H(A) = \langle r, s\rangle$ *is a non-split extension of* C *by* S_4.
(k) *The minimal degree of a faithful permutation representation of* H *is* 448; *its stabilizer is* $\langle sr^{-1}c^2sr^{-1}c^{-1}rc, cs^{-1}r^{-1}cs^{-3}c\rangle \cong A_6$.
(l) *A system of representatives* h_i *and the corresponding centralizers orders* $|C_H(h_i)|$ *of the* 31 *conjugacy classes* h_i^H *of* H *is given in Appendix* A.1.
(m) *The character table of* H *is given in Appendix* B.1.
(n) $\mathrm{Inn}(D)$ *has an elementary abelian complement of order* 8 *in* $\mathrm{Aut}(D)$.
(o) *A system of representatives* d_i *and the corresponding centralizer orders* $|C_D(d_i)|$ *of the conjugacy classes of* $D = \langle r, s\rangle$ *is given in Appendix* A.2.
(p) *The character table of* D *is given in Appendix* B.2.

Proof. All these statements have been checked by means of MAGMA from the given presentation of H. The system of representatives of the conjugacy classes of H and D have been calculated by means of Kratzer's algorithm 5.3.18 of [11], see also [9]. In particular, (k) has been obtained by calculating all core-free subgroups of H and taking the unique subgroup (up to conjugacy) realizing the smallest index among them. Since H is monolithic, its monolith being the center, a faithful permutation representation of smallest degree must be transitive. □

3. Fusion

In this section we show that any finite simple group G of *ON*-type has only one conjugacy class of involutions. By an application of Thompson's transfer lemma we obtain a new element $g \in N_G(A) - N_H(A)$ of order 3 which allows us to construct the normalizer $E = N_G(A)$, and a system of representatives of the G-conjugacy classes of elements of even order.

We owe thanks to the referee for suggesting a shortcut in the proof of assertion (a) of the following result.

Proposition 3.1. *Let* G *be a finite simple group having a* 2*-central involution* z *with centralizer* $C_G(z) = H$ *given in Lemma* 2.1. *The following statements hold*:

(a) *There exists an element* g *of order* 3 *in* $N_G(A)$, $g \notin H$ *such that* $z^g = t$, $t^g = zt$, $u^g = u$. *Such an element is unique up to conjugation in* $C = C_H(A)$.
(b) $E = N_G(A)$ *is the unique non-split extension of* $C = C_G(A) = C_H(A)$ *by* $L_3(2)$.
(c) *The automorphism group* $\mathrm{Aut}(D)$ *consists of only one* (H^*, E^*)*-double coset, where* $H^* = N_{\mathrm{Aut}(H)}(D)_{|D}$ *and* $E^* = N_{\mathrm{Aut}(E)}(D)_{|D}$.
(d) $E = \langle r, s, g\rangle$ *with set* $\mathcal{R}(E)$ *of defining relations*:

$$r^3 = g^3 = s^{16} = sr^{-1}sr^{-1}g^{-1}s^{-1}g^{-1} = (r^{-1}g^{-1}r^{-1}s^{-1})^2 = (sgs^2)^2 = 1,$$

$$(sg^{-1}r^{-1}g^{-1})^2 = (gr^{-1}s^{-1})^3 = (rs^2r^{-1}s^{-1})^2 = 1,$$

$$s^{-2}g^{-1}r^{-1}s^{-1}rs^{-2}rg = s^{-1}r^{-1}gr^{-1}s^{-1}rsgr^{-1}g^{-1} = 1.$$

(e) E *has a faithful permutation representation of degree* 112 *with stabilizer* $\langle r^{-1}sgs^2r, s^3g^{-1}s^{-1}rg^{-1}\rangle$ *whose generators have orders* 8.

(f) *A system of representatives* (e_i) *and the corresponding centralizer orders* $|C_E(e_i)|$ *of the* 18 *conjugacy classes of* e_k *in* E *are given in Appendix* A.3.
(g) *The character table of* E *is given in Appendix* B.3.

Proof. (a) Let S be the Sylow 2-subgroup of H given in Lemma 2.1(b). Using the faithful permutation representation of H described in Lemma 2.1(k) and MAGMA it has been checked that $w = rs^{-2}r^2s$ is an involution of S that does not belong to H'. Therefore $z \in H'$, $t \in H'$ and $w \in H - H'$ are representatives of all conjugacy classes of involutions of H and $|C_H(w)| = 144$ by Lemma 2.1(k).

Suppose that z and t are not conjugate in G. Then Glauberman's Z^*-Theorem implies that $z^q = w$ for some $q \in G$. Hence $C_H(w) = H \cap H^q$. Now $\frac{|H\cap H^q|}{|H'\cap H'^q|}$ divides 4 because $|H : H'| = 2$. As 16 divides $|C_H(w)|$ it follows that $|H' \cap H'^q|$ is even. Hence there is an involution $b_1 \in H' \cap H'^q$ and $v = wb_1$ is an involution in $H - H'$. Thus v and w are H-conjugate. Furthermore, $v \in H'^q$, because $w = z^q$ and b_1 belong to H'^q. Therefore $v^h = t^q$ for some $h \in H^q$. In particular, z and t are G-conjugate. This contradiction to our assumption proves that there is an x in G such that $z^x = t$, and $x \notin H$.

Let $K = C_G(t)$, then S^x is a Sylow 2-subgroup of $K = H^x$. By Lemma 2.1 $X = C_H(V) = C_H(t)$ has index 2 in S. Let Y be a Sylow 2-subgroup of K containing X. Then X is normal in Y. By Sylow's theorem there is a $y \in K$ such that $Y = S^{xy}$. Now $V = Z(X)$ char $X \lhd Y$ implies that $V \lhd Y$. Also $V^{xy} \lhd Y = S^{xy}$. By Lemma 2.1 Y has only one normal elementary abelian subgroup of order 4. Thus $V = V^{xy}$, and $xy \in N_G(V)$. Furthermore, Lemma 2.1 asserts that

$$C = C_H(A) = C_S(A) \le C_G(V) \trianglelefteq N_G(V),$$

and that C is the unique homocyclic subgroup of S of order 64. Hence $C, C^{xy} \le C_G(V) \le S$ implies that $C = C^{xy}$. Since $A = \Omega_1(C)$ it follows that $xy \in N_G(A)$. Clearly $z^{xy} = t^y = t$. In particular, $xy \notin H$. Now $C_G(A) = C_H(A) = C$. Since $D/C \cong S_4$ is a maximal subgroup of $\mathrm{Aut}(A) \cong L_3(2)$ and $xy \notin D$, it follows that $N_G(A)/C \cong L_3(2)$. Thus $E = N_G(A)$ is an extension of C by $L_3(2)$. By Lemma 2.1(j) and the Theorem of Gaschütz E is a non-split extension and $C_E(C) \le C_E(A) = C$ is 2-constrained.

Using the command `ExtensionOfSolubleGroup` implemented in MAGMA we obtain 4 extensions of C by $\mathrm{L}_3(2)$. But only one has a Sylow 2-subgroup isomorphic to the one of H. With `CosetAction` we obtain a permutation representation for this finitely presented group, identify a subgroup D_1 and check with `IsIsomorphic` that D_1 is isomorphic to D, use this isomorphism to identify r and s in D_1. Since $E/C \cong L_3(2)$ we may choose $g \in E$ such that g satisfies all 3 relations: $z^g = t, t^g = zt, u^g = u$. Let g, q be a pair of distinct elements of order 3 in E having the same action on A. Then $x = gq^{-1} \in C_G(A) = C$. By Lemma 2.1(h) C is abelian, so $a^{g^c} = a^g$ for any $c \in C$, $a \in A$. So all C-conjugates of g act as g on A. Since $|C_C(g)| = 4$ and $|C| = 64$ there are 16 such elements. MAGMA shows there are no others. As $r \in D$ acts on A by: $z^r = z, t^r = ztu, u^r = t$ it follows that their matrices $(g), (r) \in L_3(2)$ generate a Frobenius subgroup F_{21} inside $L_3(2)$. Let (s) be the matrix of s with respect to its action on A. Then $L_3(2) = \langle (r), (s), (g) \rangle$. (b) has been proved within the above argument of (a). So have (d) and (e) using MAGMA. Notice that not every element fulfilling (a) fulfills (d).

(c) An application of Kratzer's algorithm 7.1.10 of [11] proves that the number of (H^*, E^*)-double cosets in $\mathrm{Aut}(D)$ is 1. Thus by Goldschmidt's Lemma (see [4]) the amalgam (H, D, E) is unique up to isomorphism.

(f) and (g): Representatives of conjugacy classes and character tables are calculated by means of MAGMA and GAP using Kratzer's, Weller's and the second author's programs. □

Proposition 3.2. *Let G be any finite simple group of* ON*-type having a* 2*-central involution z with centralizer $H = C_G(z)$. Then the following statements hold*:

(a) *G has a unique class of involutions represented by z, denoted* $2A$.
(b) *G has* 17 *z-special H-conjugacy classes represented by*:

$$2_a, 4_a, 4_b, 6_a, 8_a, 8_b, 10_a, 12_a, 14_a, 16_a, 16_b, 16_c, 16_d, 20_a, 20_b, 28_a, 28_b.$$

(c) *Using the notation of the conjugacy classes of H and $E = N_G(A)$ as in Appendices* A.1 *and* A.3, *respectively, their fusion patterns into the special conjugacy classes of G are given in Table* 1.

Table 1.

G	$2A$			$4A$		$4B$				$8A$		$8B$		$16A$	$16B$	$16C$	$16D$
H	2_a	2_b	2_c	4_a	4_d	4_b	4_c	4_e	4_f	8_a	8_d	8_b	8_c	16_a	16_b	16_c	16_d
E	2_a	2_b		4_a		4_b	4_c			8_a		8_b		16_a	16_b	16_c	16_d

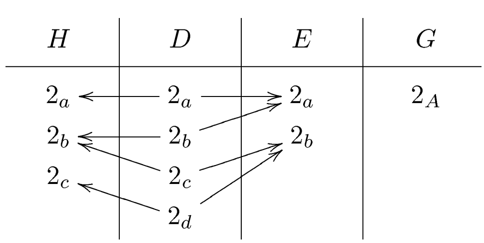

Fig. 1.

Proof. (a) The fusion of involutions can be read off from Fig. 1:
Each fusion relation has been checked using MAGMA and the Tables A.1, A.2 and A.3 classifying the conjugacy classes of H, D and E, respectively.
(b) z-special classes of H are easily read off from the conjugacy class table in Appendix A.1.

(c) The fusion graph for the other 2-elements is obtained in the same way as in (a). By [11, Theorem 7.1.8] and Proposition 3.1 the fusion of the conjugacy classes of elements of even order of H in G is uniquely determined by H. □

Lemma 3.3. *Let G be any finite simple group of* ON*-type having a 2-central involution z with centralizer $C_G(z) = H = \langle r, s, c\rangle$ given in Lemma 2.1. Then the following assertions hold*:

(a) $C_H(r) = \langle r\rangle \times \langle r_1\rangle \times \langle w, \ell\rangle$, *where* $r_1 = (s^{-1}r^{-1})^2c^{-1}r^{-1}c$ *has order* 3 *and* $D_1 = \langle \ell, w\rangle$ *is a dihedral group of order* 8 *generated by the involution* $w = rsrc^{-2}sc^{-1}r^{-1}s^{-1}$ *and* $\ell = (rc)^3$ *of order* 4 *satisfying* $\ell^w = \ell^{-1}$.
(b) $N_1 = N_H(r) = \langle C_H(r), n_1\rangle$, *where* $n_1 = (s^{-1}c[c^2, s])^2$ *is an involution satisfying the relations*: $r^{n_1} = r^2$, $r_1^{n_1} = r_1^2$, $[x, n_1] = 1$ *for all* $x \in D_1$.
(c) $S_2 = D_1 \times \langle n_1\rangle$ *is a Sylow* 2*-subgroup of* N_1.
(d) $R = \langle r\rangle \times \langle r_1\rangle$ *is a Sylow* 3*-subgroup of* H.
(e) $N_H(R) = (R \times D_1) : Q_8$, *where the quaternion subgroup* $Q_8 = \langle q_1, q_2\rangle$ *is generated by* $q_1 = s^3r^{-1}c^{-1}rc^{-1}s^{-1}$ *and* $q_2 = c^{-1}s^{-1}cr^{-1}c^{-1}r^{-1}c^{-1}$, *satisfying the following relations*:

$$r^3 = r_1^3 = 1, \ell^4 = 1,\ w^2 = 1,\ q_1^4 = 1,\ q_2^4 = 1,\ (\ell^{-1}w)^2 = 1,\ [\ell, q_1] = 1,\ q_2^{-1}q_1^2q_2^{-1} = 1,\ \ell^{-1}q_2^{-1}\ell^{-1}q_2 = 1,\ q_1^{-1}q_2^{-1}q_1q_2^{-1} = 1,\ wq_2^{-1}\ell^{-1}wq_2 = 1,\ q_1^{-1}w\ell^2q_1w = 1, r^{q_1} = r_1, r_1^{q_1} = r^2, r^{q_2} = rr_1^2, r_1^{q_2} = (rr_1)^2.$$

Proof. Using the faithful permutation representation of H given in Lemma 2.1 and MAGMA all these statements have been deduced from Lemma 2.1 computationally. □

Proposition 3.4. *Let G be a simple* ON*-type group having a* 2*-central involution z such that $C_G(z) = \langle r, s, c\rangle$. Let $N_1 = N_H(r) = \langle r, r_1, w, \ell, n_1\rangle$ be as in Lemma* 3.3 *and let $R = \langle r, r_1\rangle$. Then the following assertions hold*:

(a) $C_G(r) \cong R \times A_6 \cong C_G(R)$.
(b) *There is an element f_1 of order* 5 *in G such that $N_G(r) = \langle r, r_1, l, n_1, w, f_1\rangle$ is isomorphic to $(R \times A_6) \cdot 2$ and has the following presentation*:

$$r^3 = 1,\ r_1^3 = 1,\ n_1^2 = 1,\ \ell^4 = 1,\ w^2 = 1,\ f_1^5 = 1,\ [r, r_1] = 1,\ (r^{-1}n_1)^2 = 1,\ (r_1^{-1}n_1)^2 = 1,\ [r, \ell] = 1,\ [r_1, \ell] = 1,\ n_1\ell^{-1}n_1\ell = 1,\ r^{-1}wrw = 1,\ r_1^{-1}wr_1w = 1,\ (n_1w)^2 = 1,\ (\ell^{-1}w)^2 = 1,\ [r, f_1] = 1,\ [r_1, f_1] = 1,\ n_1f_1^{-1}n_1f_1 = 1,\ (wf_1^{-1})^2 = 1,\ (f_1^{-1}\ell^{-1})^3 = 1,\ \ell^{-2}f_1^{-1}\ell^2f_1^{-1}\ell^2f_1^{-1} = 1,\ \ell^{-1}f_1\ell^{-1}f_1^{-1}w\ell^{-1}f_1^{-2}\ell f_1 = 1.$$

(c) *G has* 3 *r-special conjugacy classes represented by the following classes of the character table of $N = N_G(r)$ given in Appendix* B.4:

$$3_a,\ 15_a,\ 15_b.$$

(d) *A system of representatives n_i and the corresponding centralizer orders $|C_N(n_i)|$ of the* 42 *conjugacy classes n_i^N of $N = N_G(r)$ are given in Appendix* A.5.
(e) *$n_1 \in N - C_G(r)$ inverts r and centralizes the subgroup $U_1 = \langle \ell, w, f_1\rangle \cong A_6$ of N.*

Proof. (a) By Lemma 3.3 $C_H(r) = R \times D_1$, where R is elementary abelian of order 9 and $D_1 = \langle \ell, w \rangle$ is a dihedral Sylow 2-subgroup of $C = C_G(r)$.

Let j be any involution of $C_G(r)$. By Proposition 3.2(a) there is an $x \in G$ such that $z = j^x$. Hence

$$r^x \in C_G(j^x) = C_G(z) = H.$$

By Lemma 2.1(l) H has a unique conjugacy class of elements of order 3 represented by r. Hence there is an $h \in H$ such that $r^{xh} = r$ and $z = z^h = j^{xh}$. Therefore $xh \in C_G(r)$. Thus all involutions of $C = C_G(r)$ are conjugate in C.

Let $Q = O(C)$. Then $C/Q \neq D_1$, because D_1 has 3 conjugacy classes of involutions. Now the Gorenstein–Walter theorem (see [5]) asserts that

$$C/Q \in \{A_6, A_7, L_2(7)\}.$$

In the case of A_7 the 2-central involution would have a centralizer in A_7 of order divisible by 3. Hence $|H| = |C_G(z)|$ would be divisible by 3^3, a contradiction to the order of H. Thus $C/Q \in \{A_6, L_2(7)\}$.

Suppose that $C/Q \cong L_2(7)$. By Lemma 3.3 $|N_H(r)/C_H(r)| = 2$, and there is an involution $n_1 \in N_H(r)$ satisfying $r^{n_1} = r^2$. If the residue class of n_1 in C/Q were operating like an outer automorphism of $L_2(7)$, then the Sylow 2-subgroup of $N_G(r)$ would be a dihedral group of order 16. Since $N = N_G(r)$ and $N_H(r)$ have conjugate Sylow 2-subgroups, this is impossible by Lemma 3.3(c).

Hence we may assume that n_1 centralizes the normal subgroup $L_2(7)$ of N/Q. If the residue class of n_1 were not generating a direct factor of $L_2(7)$ in N/Q, then $N/Q = SL_2(7)$. In particular, N would have a generalized quaternion Sylow 2-subgroup of order 16 as a Sylow 2-subgroup, which is a contradiction to Lemma 3.3(c). Therefore $N/Q = \langle n_1 \rangle \times L_2(7)$. As G has a unique conjugacy class of involutions n_1 is conjugate to z, and $L_2(7)$ is a subgroup of H up to conjugation.

Another application of MAGMA yields that H has a unique conjugacy class of subgroups $U \cong L_2(7)$ and that their dihedral Sylow 2-subgroups D_2 of order 8 contain a unique class of elements v of order 4 which are H-conjugate to $(rs)^2 \in 4_c$. By Lemma 2.1(l) $\ell = (rc)^3 \in D_1$ belongs to 4_a Proposition 3.2(c) implies that ℓ and $(rs)^2$ are not conjugate in G. Hence the Sylow 2-subgroup D_1 of $C = C_G(r)$ and the Sylow 2-subgroup D_2 of $L_2(7) \leq C$ cannot be conjugate in C, a contradiction to Sylow's Theorem. Therefore $C/Q = A_6$.

Applying the Brauer–Wielandt theorem to the Klein 4-subgroup $V = \langle z, w \rangle$ of D_1 it follows that

$$|Q||C_Q(V)|^2 = |C_Q(z)|^3, \tag{3.1}$$

because C has a unique class of involutions. Furthermore, $r \in C_Q(z)$, and $C_Q(z) \leq H \cap C_G(r) = C_H(r)$. Since $|C_H(r)| = 2^3 3^2$ by Lemma 2.1, we get that $|C_Q(z)| \in \{3, 9\}$.

Suppose that $|C_Q(z)| = 3$. As $r \in C_Q(z) \cap C_Q(w)$ it follows from (3.1) that $|C_Q(V)| = |C_Q(z)| = |Q| = 3$. Hence $Q = \langle r \rangle$, and $C/Q \cong A_6$. Therefore either $C \cong 3A_6$ or $C \cong \langle r \rangle \times A_6$.

Let $C_1 = C_G(R)$. As $C_H(r) = R \times D_1 = C_H(R)$ also C_1 has a dihedral Sylow 2-subgroup D_1. Let $W = O(C_1)$. Then the Gorenstein–Walter theorem asserts that

$$C_1/W \in \{L_2(7), A_6, A_7, D_1\}.$$

As $C_1 \leq C$ and 7 does not divide $|C| = 3|A_6|$, the simple groups $L_2(7)$ and A_7 are not involved in C_1. Clearly, $R \leq W$. If $C_1/W \cong A_6$, then

$$3|A_6| = |C| \geq |C_1| = |W||A_6| \geq 9|A_6|,$$

a contradiction. Thus $C_1/W \cong D_8$, the dihedral group of order 8.

Applying now the Brauer–Wielandt theorem again to the Klein 4-subgroup $V = \langle z, w\rangle$ of the Sylow 2-subgroup D_1 of C_1 it follows that

$$|W||C_W(V)|^2 = |C_W(z)||C_W(w)|^2,$$

because the involutions w and wz are conjugate in D_1. As $|C_W(z)| = |R| \leq |C_W(w)|$, we get $|C_W(V)| = |R|$. Therefore $|W| = |R|^{-1}|C_W(w)|^2$. Since $C_1 \leq C$ and $|C|$ has a Sylow 3-subgroup of order 27 we see that $W = R$. Hence

$$C_1 = C_G(R) \cong W \times D_8 \cong R \times D_1.$$

In particular, $|R| = 9$ divides the order of $C_G(D_1)$. However, if $C \in \{3A_6, 3 \times A_6\}$, then $|C_C(D_1)| = 6$. This contradiction proves that $Q = R$, and $C/R \cong A_6$. Hence $C_G(R)/R \cong A_6$. By Lemma 3.3(e) $N_H(R) = (R \times D_8) : Q_8$. Furthermore, Q_8 acts irreducibly on R, because n_1 inverts both generators r and r_1 of R. Since the Schur multiplicator of A_6 is cyclic of order 6 this implies that $C_G(R) \cong R \times A_6$, and so

$$C = C_G(r) = C_G(R) \cong R \times A_6,$$

which completes the proof of (a). The remaining assertions are now easy to verify computationally by means of MAGMA. In particular, statement (e) follows from the table of representatives of the conjugacy classes given in Appendix A.5. □

4. Embedding of the Janko Group J_1 into Simple ON-Type Groups

In this section we show that each finite simple group G of ON-type having a 2-central involution z with centralizer $C_G(z) = H = \langle c, r, s\rangle$ has a subgroup $J = \langle r, f, g\rangle$ isomorphic to the small Janko group J_1 of order $|\mathsf{J}_1| = 175560$, where g is the element of G defined in Proposition 3.1, and f is a suitable element of order 5 in H.

Lemma 4.1. *Let G be a finite simple group of ON-type with a 2-central involution z having centralizer $C_G(z) = H = \langle r, s, c\rangle$. Let $A = \langle z, t, u\rangle$ be the unique elementary abelian normal subgroup of the finite Sylow 2-subgroup S of H defined in Lemma 2.1. Let $g \in G$ be the element of order 3 defined in Proposition 3.1. Then the following assertions hold*:

(a) *The element $f = r^{-1}c^{-1}r^{-1}c^{-2}rc^{-1}r^{-1} \in H$ has order 5.*

(b) $H_1 = \langle z, t, u, r, f\rangle$ *is the only subgroup of* H *of shape* $\langle z\rangle \times A_5$ *containing the subgroup* $\langle A, r\rangle$.

(c) H_1 *has the following set* $\mathcal{R}(H_1)$ *of defining relations*:
$t^2 = u^2 = r^3 = f^5 = (zt)^2 = (zu)^2 = (tu)^2 = 1$,
$zr^{-1}zr = tr^{-1}ur = zf^{-1}zf = (tf^{-1})^2 = 1$,
$r^{-1}utzru = f^{-1}r^{-1}utfr^{-1} = f^{-2}utf^{-1}r^{-1} = 1$.

(d) $E_1 = \langle r, g\rangle$ *is the split extension of the elementary abelian group* $A = \langle z, t, u\rangle$ *with a Frobenius group* $F_{21} = \langle [r, r^g]\rangle : \langle r\rangle \cong 7 : 3$. *Moreover,* $C_A(r) = \langle z\rangle$, $C_A(g) = \langle u\rangle$.

(e) E_1 *has the following set* $\mathcal{R}(E_1)$ *of defining relations*:
$t^2 = u^2 = r^3 = g^3 = (zt)^2 = (zu)^2 = (tu)^2 = 1$,
$zr^{-1}zr = tr^{-1}ur = tg^{-1}zg = ug^{-1}ug = 1$,
$g^{-1}tzgz = r^{-1}utzru = r^{-1}gr^{-1}zgrg^{-1} = 1$.

Proof. (a) and (b) have been checked by means of MAGMA and the faithful permutation representation of H described in Lemma 2.1(k).

(c) The presentation of H_1 has been calculated by means of MAGMA.

(d) $E_1 = \langle z, t, u, r, g\rangle \le E = N_G(A) = \langle r, s, g\rangle$ by Proposition 3.1. Using the faithful permutation representation of E given in Proposition 3.1(f) and MAGMA it follows that E_1 is generated by r and g, and that $A = \langle z, t, u\rangle$ is a normal Sylow 2-subgroup of E_1 having a Frobenius group $F_{21} = \langle r, r^g\rangle$ as complement, where

$$z = rg^{-1}rgr^{-1}g^{-1}, \quad u = (rg^{-1})^2rgr, \quad t = z^g.$$

The remaining statements are now trivial.

(e) The presentation of E_1 has been determined by means of MAGMA. □

Proposition 4.2. *Each finite simple group* G *of* ON*-type has a simple subgroup* J *isomorphic to the Janko's small sporadic simple group* J_1 *of order* $|\mathsf{J}_1| = 175560$. *Furthermore,* $J = \langle r, f, g\rangle$ *has the following set* $\mathcal{R}(J)$ *of defining relations*:

$$f^5 = r^3 = g^3 = (rf)^3 = (f^2r^{-1})^2 = rgfr^{-1}f^{-1}grg = 1,$$
$$rg^{-1}r^{-1}frf^{-1}gr^{-1}g^{-1} = rgf^{-2}g^{-1}r^{-1}fgrf^{-1}g^{-1}f^{-1} = 1.$$

Proof. Let z be a 2-central involution with centralizer $C_G(z) = H = \langle r, s, c\rangle$ defined in Lemma 2.1. Let S be the fixed Sylow 2-subgroup of H chosen in Lemma 2.1. Let $A = \langle z, t, u\rangle$ be the unique maximal elementary abelian normal subgroup of S. Proposition 3.1 asserts that there is an element $g \in N_G(A) - N_H(A)$ of order 3 such that $z^g = t$, $t^g = zt$ and $u^g = u$.

By Lemma 4.1(a) and (b) the element $f = r^{-1}c^{-1}r^{-1}c^{-2}rc^{-1}r^{-1} \in H$ has order 5 and the subgroup $H_1 = \langle z, t, u, r, f\rangle \cong \langle z\rangle \times A_5$. Let $\mathcal{R}(H_1)$ be the set of defining relations of H given in Lemma 4.1(c). Let $\mathcal{R}(E_1)$ be the set of defining relations of the subgroup $E_1 = \langle z, t, u, r, g\rangle \cong A : F_{21}$ of G given in Lemma 4.1(e).

Keeping the notations of Proposition 3.4 we know that in any group G of ON-type

$$N := N_G(r) = [\langle r, r_1\rangle : \langle n_1\rangle] \times \langle l, w, f_1\rangle,$$

where $X = \langle l, w, f_1\rangle \cong A_6$ and f_1 is of order 5.

Using MAGMA and the faithful permutation representation of H given in Lemma 2.1(k) one can see that $h = (frs^2)^2 s^{12} \in H$ has order 8 and satisfies the following equations $n_1^h = zt$ and $r^h = [r^2, f]$. Hence $n_1^{hg} = z$ by Proposition 3.1. Therefore

$$X^{hg} \leq C_G(n_1^{hg}) = C_G(z) = H,$$
$$N^{hg} = N_G(r^{hg}) = [\langle r^{hg}, r_1^{hg}\rangle : \langle z\rangle] \times X^{hg}.$$

Now using MAGMA again we see that $w^h \in D = N_H(A)$. As $l^h = l$ it follows that the dihedral Sylow 2-subgroup $Y = \langle l, w\rangle$ of X is mapped under conjugation by hg to

$$Y^{hg} = \langle l, w^h\rangle^g \leq H \cap E = D,$$

because $g \in E = N_G(A)$ by Proposition 3.1.

Another application of MAGMA yields then that there are exactly 2 subgroups $L_i \cong \langle z\rangle \times A_6$ of H containing Y^{hg}. Hence $X^{hg} \in \{L_1, L_2\}$. Furthermore it has been checked that $f \in L_1$ and $f^u \in L_2$ up to choosing the indices of the L_i. Hence exactly one of the following relations holds:

$$[r^{hg}, f] = [[r^2, f]^g, f] = 1, \tag{4.1}$$
$$[r^{hg}, f^u] = [[r^2, f]^g, f^u] = 1. \tag{4.2}$$

Using the Todd–Coxeter Algorithm implemented in MAGMA it has been checked that the subgroup $J = \langle H_1, E_1\rangle = \langle z, u, t, r, f, g\rangle$ of G having the set

$$\mathcal{R}(J) = \mathcal{R}(H_1) \cup \mathcal{R}(E_1) \cup (4.2)$$

as defining set of relations with respect to its generating set $\{z, u, t, r, f, g\}$ is the identity group. This contradiction shows that $f \in L_1 = X^{hg}$. In particular, the relation (4.1) holds in J. Another application of the Todd–Coxeter Algorithm now shows that the subgroup $J = \langle H_1, E_1\rangle = \langle z, u, t, r, f, g\rangle$ of G with the set

$$\mathcal{R}(J) = \mathcal{R}(H_1) \cup \mathcal{R}(E_1) \cup (4.1)$$

as defining relations has a faithful permutation representation of degree $|J : H_1| = 1463$ with stabilizer H_1. Using this new permutation representation we see that J is simple and has order 175560. Therefore $J \cong \mathsf{J}_1$ by [11, Theorem 9.4.2], see also [8]. The short presentation of J given in the statement of this proposition has been obtained by MAGMA and the above permutation representation. $\square$

5. Main Theorem

In this section it is proved that all finite simple groups of ON-type are isomorphic by showing that they all have the same presentation. Furthermore, this presentation also yields an existence proof of a finite simple group G of ON-type. It follows that the O'Nan group G has a faithful permutation representation of degree 2.624.832 with stabilizer J isomorphic to Janko's smallest sporadic simple group J_1.

Lemma 5.1. *Let G be a finite simple group of ON-type having a 2-central involution z with centralizer $C_G(z) = H$. Let S be the fixed Sylow 2-subgroup of H and let $A = \langle z, t, u\rangle$ be the unique maximal elementary abelian normal subgroup of S given in Lemma 2.1. Then the following assertions hold:*

(a) *G has a unique conjugacy class of involutions.*
(b) *$G = \langle H, N_G(A)\rangle$.*

Proof. (a) holds by Proposition 3.2(a).

(b) By Lemma 2.1 the Sylow 2-subgroup S of H has derived length 3. Thus a simple group G of ON-type cannot be isomorphic to $PSL_2(q)$, $Sz(q)$ or $PSU_3(q)$, where q is a power of 2. Therefore the Bender–Suzuki theorem stated as Theorem 4.2 in Suzuki [17, p. 392] implies that $G = \langle H, N_G(S), C_G(t) \mid t \in I(H)\rangle$, where $I(H)$ denotes the set of all involutions t of H. Clearly, A is a characteristic subgroup of S. Hence $N_G(S) \le N_G(A) = E$. By (a) G has a unique conjugacy class of involutions. Now the proof of Proposition 3.2 implies that all involutions $t \in I(H)$ are fused to z in the subgroup $\langle H, E\rangle$ of G. Thus

$$G = \langle H, N_G(S), C_G(t) \mid t \in I(H)\rangle \le \langle H, E\rangle \le G,$$

which completes the proof. □

Theorem 5.2. *Let G be a finite simple group of ON-type with a 2-central involution z having centralizer $C_G(z) = H = \langle r, s, c\rangle$ and fixed Sylow 2-subgroup $S \le H$ defined in Lemma 2.1. Let $A = \langle z, t, u\rangle$ the unique maximal elementary abelian normal subgroup of S. Then the following assertions hold:*

(a) *There exists an element $g \in N_G(A) - N_H(A)$ of order 3 such that $z^g = t$, $t^g = zt$ and $u^g = u$, and $E = N_G(A) = \langle r, s, g\rangle$.*
(b) *The elements $f = r^{-1}c^{-1}r^{-1}c^{-2}rc^{-1}r^{-1}$ and $c = s^2fr^2frs^2r^2$ of H have order 5, and*

$$H = \langle r, s, c\rangle = \langle r, s, f\rangle.$$

(c) *$G = \langle r, s, f, g\rangle$ has the set*

$$\mathcal{R}(G) = \mathcal{R}_1(H) \cup \mathcal{R}(J) \cup \mathcal{R}(E)$$

as a set of defining relations, where $H = \langle r, s, f \mid \mathcal{R}_1(H)\rangle$. *Leaving out repetitions* $\mathcal{R}(G)$ *is described by the following relations*:
$r^3 = s^{16} = f^5 = g^3 = 1,$

$r^{-1}g^{-1}f^{-1}rgf^2g^{-1}r^{-1}fgf = s^{-1}r^{-1}gr^{-1}s^{-1}rsgr^{-1}g^{-1} = 1,$
$frf^{-1}s^{-2}r^{-1}s^2r^{-1}s^{-2} = (r^{-1}g^{-1}r^{-1}s^{-1})^2 = s^{-1}f^{-1}s^4f^{-1}s^{-3} = 1,$
$(rf^{-1}sf^{-1}s^{-1})^2 = (r^{-1}s^2rs^{-1})^2, f^{-2}gf^{-1}g^{-1}f^2r^{-1}gfg^{-1}r = 1,$
$(gr^{-1}s^{-1})^3 = (sg^{-1}r^{-1}g^{-1})^2 = (sgs^2)^2 = (s^{-1}f^{-1})^4 = 1,$
$r^{-1}sfrs^{-1}r^{-1}sf^2s^{-1}f^{-1} = (f^{-1}r^{-1})^3, (rf^{-2})^2 = 1,$
$(sf^{-1})^4 = 1, sr^{-1}sr^{-1}g^{-1}s^{-1}g^{-1} = (fr^{-1}f)^2 = (rs^2r^{-1}s^{-1})^2 = 1,$
$s^{-2}g^{-1}r^{-1}s^{-1}rs^{-2}rg = fr^{-1}f^{-1}r^{-1}s^{-1}r^{-1}fr^{-1}f^{-1}s = 1,$
$rgfr^{-1}f^{-1}grg = rg^{-1}fr^{-1}f^{-1}rgr^{-1}g = 1.$

(d) *The simple subgroup* $J = \langle r, f, g\rangle$ *of order* 175560 *has index* $|G : J| = 2.624.832$.
(e) G *is uniquely determined by* H *up to isomorphism and has order*

$$|G| = 2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31.$$

(f) *The finitely presented group* $G = \langle r, s, f, g\rangle$ *with set* $\mathcal{R}(G)$ *of defining relations is a simple group with* 2*-central involution* $z = s^8$ *such that* $C_G(z) = \langle r, s, f\rangle \cong H$.

Proof. Assertion (a) holds by Proposition 3.1(a) and (d). (b) Lemma 4.1(a) states that $f = r^{-1}c^{-1}r^{-1}c^{-2}rc^{-1}r^{-1} \in H$ has order 5. Using MAGMA and the faithful permutation representation of H given in Lemma 2.1(k) we checked that

$$H = \langle r, s, c\rangle = \langle r, s, f\rangle$$

and that $c = s^2fr^2frs^2r^2$.

(c) Furthermore, it has been verified that the following set $\mathcal{R}_1(H)$ is a defining set of relations of H with respect to $\{r, s, f\}$:

$r^3 = s^{16} = f^5 = (f^{-1}r^{-1})^3 = (rf^{-2})^2 = (s^{-1}f^{-1})^4 = (sf^{-1})^4 = 1,$
$(rs^2r^{-1}s^{-1})^2 = fr^{-1}f^{-1}r^{-1}s^{-1}r^{-1}fr^{-1}f^{-1}s = (r^{-1}s^2rs^{-1})^2 = 1,$
$(rf^{-1}sf^{-1}s^{-1})^2 = s^{-1}f^{-1}s^4f^{-1}s^{-3} = r^{-1}sfrs^{-1}r^{-1}sf^2s^{-1}f^{-1} = 1,$
$frf^{-1}s^{-2}r^{-1}s^2r^{-1}s^{-2} = 1.$

By Proposition 4.2 the subgroup $J = \langle r, f, g\rangle$ has the following set $\mathcal{R}(J)$ of defining relations:

$f^5 = r^3 = g^3 = (fr^{-1}f)^2 = rgfr^{-1}f^{-1}grg = rg^{-1}fr^{-1}f^{-1}rgr^{-1}g = 1,$
$f^{-2}gf^{-1}g^{-1}f^2r^{-1}gfg^{-1}r = r^{-1}g^{-1}f^{-1}rgf^2g^{-1}r^{-1}fgf = 1.$

Proposition 3.1 asserts that $E = N_G(A) = \langle r, s, g\rangle$ has the following set $\mathcal{R}(E)$ of defining relations:

$r^3 = g^3 = s^{16} = 1,\ sr^{-1}sr^{-1}g^{-1}s^{-1}g^{-1} = (r^{-1}g^{-1}r^{-1}s^{-1})^2 = (sgs^2)^2 = 1,$
$(sg^{-1}r^{-1}g^{-1})^2 = (gr^{-1}s^{-1})^3 = (rs^2r^{-1}s^{-1})^2 = 1,$
$s^{-2}g^{-1}r^{-1}s^{-1}rs^{-2}rg = s^{-1}r^{-1}gr^{-1}s^{-1}rsgr^{-1}g^{-}1 = 1.$

Hence $G_1 = \langle H, J, E\rangle = \langle r, s, c, f, g\rangle = \langle r, s, f, g\rangle$ is a subgroup of G with the set

$$\mathcal{R}(G_1) = \mathcal{R}_1(H) \cup \mathcal{R}(J) \cup \mathcal{R}(E)$$

as set of relations. As J is a subgroup of G_1, we applied the Todd–Coxeter algorithm implemented in MAGMA and asked it to count the residue classes of J in G_1. After a few minutes it returned:

$$|G_1 : J| = 2.624.832.$$

Hence $\mathcal{R}(G_1)$ is a set of defining relation of G_1. Now, Lemma 5.1 asserts that

$$G = \langle H, E\rangle \le \langle H, J, E\rangle = G_1,$$

which completes the proof of (c).

Assertions (d) and (e) are immediate consequences of Proposition 4.2 and statement (c).

(f) Now let $G = \langle r, s, f, g\rangle$ be the finitely presented group with set $\mathcal{R}(G)$ of defining relations. By (b), (e) and Lemma 2.1 $z = s^8$ is a 2-central involution with centralizer $C_G(z) \ge \langle r, s, f\rangle \cong H$. Using the faithful permutation representation of G with stabilizer $J = \langle r, f, g\rangle$ and MAGMA we see that the involution z has 1344 fixed points on the permutation module $(1_J)^G$. By Propositions 4.2 and 3.2 both groups J and G have a unique conjugacy class of involutions represented by z, respectively. Hence [11, Proposition 2.6.6] asserts that $1344 = \frac{C_G(z)}{C_J(z)}$. Thus $|C_G(z)| = 161280$, and $C_G(z) = \langle r, s, f\rangle \cong H$.

Let $X \ne 1$ be a proper normal subgroup of G. If $|X|$ is even, then $z^G \subseteq X$ by Proposition 3.2. As $H := C_G(z) = \langle r, s, f\rangle$ we can apply all statements of Lemma 2.1. Therefore $y = (sc^2)^3 \in H - H'$ by Lemma 2.1(l), and H has a faithful permutation representation of degree 448 by Lemma 2.1(k). Using it together with MAGMA it follows that $H = \langle y^H\rangle$. Hence $H \le X$. In particular, $r \in X$. By Proposition 3.1 we know that $E = \langle r, s, g\rangle$ has a unique conjugacy class r^E of elements of order 3. Hence $g \in r^E \le X$, and $G = \langle H, g\rangle = X$. This contradiction shows that $|X|$ is odd. By Lemma 2.1(l) $W = \langle z, t\rangle$ with $t := (rs)^4$ is a Klein four group and $C_X(z) = X \cap C_G(z) = 1$. As W acts on X by conjugation the Brauer–Wielandt Lemma asserts that

$$|X||C_X(W)|^2 = |C_X(z)||C_X(t)||C_X(zt)| = |C_X(z)|^3,$$

because all involutions are conjugate in G. Now $C_X(W) \le X \cap C_H(t) = 1$. So $|X| = |H \cap X|^3$. But $H \cap X \le O(H) = 1$. Thus $X = 1$, and the proof is complete. □

Lemma 5.3. *The normalizers $N_G(p)$ of the Sylow p-subgroups of G for the primes $11, 19, 31$ are Frobenius groups of orders $11 \cdot 10, 19 \cdot 6, 31 \cdot 15$, respectively.*

Proof. Using Theorem 5.2 and Sylow's Theorem it is easy to get the orders of $N_G(p)$. □

6. A Permutation Representation of Degree 122.760

In this section a small faithful permutation representation of the O'Nan group G with stabilizer $L \cong L_3(7) : 2$ is determined. It is used to get a complete classification of all conjugacy classes of $G = \langle r, s, f, g\rangle$ in terms of short words in the generators of G. Thus we are able to determine the fusion of the conjugacy classes of the local subgroups and the Janko subgroup J of G found in the earlier sections in G. Furthermore, the classification of the conjugacy classes of G and Theorem 5.2 allow us to apply Wilson's concept of standard generators [18] to get another subgroup $L_2 \cong L_3(7) : 2$ which is not conjugate to L in G.

Proposition 6.1. *Let $G = \langle r, s, f, g\rangle = \langle c, r, s, f, g\rangle$ be the finitely presented group with set $\mathcal{R}(G)$ of defining relations given in Theorem 5.2. Let*

$$\begin{aligned}
w &= rsrc^{-2}sc^{-1}r^{-1}s^{-1},\\
p_1 &= rs^3c^{-1}r^{-1}s^{-1}c^{-1},\\
p_2 &= s^2c^{-1}rc^2s^{-1},\\
n_1 &= (s^{-1}c[c^2, s])^2,\\
x_1 &= [r^{-1}, f]^{g^2(rfr)^{-1}},\\
a &= p_1p_2p_1x_1^{-1}p_1x_1^{-1}n_1p_1p_2^{-1}x_1,\\
d &= p_2^{-1}x_1^{-1}p_1p_2^{-1}p_1x_1^{-1}p_1n_1p_2^{-1}x_1,\\
a_1 &= rfr^{-1}s^{-4}r,\\
d_1 &= g^{-1}rf^{-1}r^{-1}fg,\\
q_1 &= s^{-1}grs^2r^{-1}g.
\end{aligned}$$

Then the following assertions hold:

(a) *$Y = N_G(f) = \langle a_1, d_1, q_1\rangle$ admits the following presentation*:
$d_1^3 = q_1^4 = q_1a_1^3q_1^{-1}a_1^{-1} = a_1d_1a_1^{-2}d_1a_1 = a_1q_1^{-2}a_1^2q_1^2a_1 = d_1q_1^{-2}d_1q_1^2 = 1.$

(b) *A system of representatives y_i and corresponding centralizer orders $|C_Y(y_i)|$ of the* 18 *conjugacy classes y_i^Y of Y is given in Appendix* A.6.

(c) *The character table of Y is given in Appendix* B.5.

(d) *$R = \langle a, d\rangle \cong N_G(7_A) \cong 7^{1+2} : (3 \times D_8)$, where D_8 is a dihedral subgroup of order* 8.

(e) *A system of representatives r_i and corresponding centralizer orders $|C_R(r_i)|$ of the* 24 *conjugacy classes r_i^R of R is given in Appendix* A.8.

(f) *The character table of R is given in Appendix* B.6.

(g) *$L = \langle p_1, p_2, x_1, n_1\rangle$ is a subgroup of G with the following set $\mathcal{R}(L)$ of defining relations*:

$$\begin{aligned}
&p_1^4 = p_2^6 = x_1^3 = n_1^2 = 1,\\
&p_1^{-1}p_2^{-1}p_1^2p_2p_1^{-1} = p_1^{-1}n_1p_1^2n_1p_1^{-1} = (p_1^{-1}n_1p_2^{-1})^2 = 1,\\
&x_1^{-1}n_1p_1^{-2}x_1p_1^2n_1 = n_1x_1^{-1}n_1x_1^{-1}p_1^2x_1^{-1} = x_1n_1x_1^{-1}p_2^{-1}p_1p_2n_1p_2 = 1,\\
&n_1p_2^{-3}x_1^{-1}p_2^3n_1x_1^{-1} = (p_2^{-1}p_1^{-1}p_2^{-1})^3 = n_1x_1^{-1}p_2^2p_1^{-1}p_2^{-1}n_1p_1x_1 = 1,\\
&x_1^{-1}p_1^{-1}x_1^{-1}p_1^{-1}x_1^{-1}p_2x_1p_2x_1p_2^{-1} = 1.
\end{aligned}$$

(h) $L \cong L_3(7) : 2$.
(i) *A system of representatives* ℓ_i *and the corresponding centralizer order* $|C_L(\ell_i)|$ *of the* 26 *conjugacy classes* ℓ_i^L *of* L *is given in Appendix* A.7.
(j) *The character table of* L *coincides with the one of* $\mathrm{L}_3(7) : 2$ *given in* [2, *p.* 51].
(k) *The subgroup* $T = \langle t_i \mid 1 \leq i \leq 6 \rangle$ *of* L *with generators*:

$$t_1 = p_1 n_1 p_2^{-1} x_1 p_2^{-2} p_1 x_1^{-1} p_1^{-1} n_1,$$
$$t_2 = p_1 x_1 p_1 p_2^{-1} x_1 p_1 p_2 p_1 p_2^{-1} p_1^{-1} x_1,$$
$$t_3 = p_1 x_1 p_2^{-1} n_1 x_1^{-1} p_1 x_1^{-1} n_1 p_2 p_1,$$
$$t_4 = n_1 p_1 x_1 p_2^{-1} n_1 p_2 n_1 x_1 p_1 x_1^{-1},$$
$$t_5 = p_2^{-1} n_1 p_1 x_1 p_1 p_2 p_1 x_1 p_2^{-1} n_1,$$
$$t_6 = x_1 p_2 n_1 p_1 x_1^{-1} p_1 x_1 p_2 p_1^{-1} n_1,$$

is the stabilizer of a faithful permutation representation of L *having degree* 456.

Proof. (g) It is well known that $L' = PSL_3(7)$ has a unique conjugacy class of involutions z with centralizer $H_2 = C_{L'}(z) \cong SL_2(7) \cdot 2$. Using the faithful permutation representation of H described in Lemma 2.1(k) and MAGMA we found the two generators p_1 and p_2 of H_2 with orders 4 and 6, respectively. Another application of MAGMA yields that $H_3 = H_2\langle n_1 \rangle$ is isomorphic to the centralizer of a 2-central involution of $L = L_3(7) : 2$. By Theorem 5.2 $G = \langle r, s, f, g \rangle$ has a faithful permutation representation of degree 2.624.832 with stabilizer $J = \langle r, f, g \rangle$. Using this faithful permutation representation and MAGMA it has been checked that the subgroup $L = \langle p_1, p_2, x_1, n_1 \rangle$ of G has index 122.760. Hence $|L| = 2^6 \cdot 3^2 \cdot 7^3 \cdot 19$ by Theorem 5.2. We now use the faithful permutation representation of G with stabilizer L and MAGMA to get the given presentation of L.

(k) It also has been used to check that $T = \langle t_i \mid 1 \leq i \leq 6 \rangle$ is a subgroup of L with index $|L : T| = 456$. It affords a faithful permutation representation of L.

(h), (i) and (j): This small permutation representation of L and MAGMA were applied to verify that $L \cong L_3(7) : 2$. Using then also Kratzer's algorithm 5.3.18 of [11] the representatives of the conjugacy classes of L stated in Table A.7 were calculated. The character table of L was then obtained by means of MAGMA.

(d), (e) and (f): Using again the small permutation representation of L it follows that $N_L(d^4) = \langle a, d \rangle = R \cong N_L(7_A) \cong 7^{1+2} : (3 \times D_8)$. By Theorem 5.2(e) and Sylow's Theorem we have $N_G(7_A) = N_L(7_A)$. Assertions (e) and (f) have been proved by means of the small permutation representation of L, MAGMA and Kratzer's algorithm mentioned above.

(a), (b) and (c): These 3 assertions have been proved by means of the faithful permutation representation $(1_L)^G$ constructed in (g), MAGMA and Kratzer's algorithm mentioned above. □

Corollary 6.2. *Let* $G = \langle r, s, f, g \rangle$ *be the finitely presented O'Nan group with set* $\mathcal{R}(G)$ *of defining relations given in Theorem* 5.2. *Then* G *has* 30 *conjugacy classes.*

In Table A.9 *a complete set of their representatives are given in terms of short words in the generators of* G.

Proof. The representatives x and the orders $|C_G(x)|$ given in Table A.9 have been calculated by means of the faithful permutation representation of G constructed in Proposition 6.1, MAGMA and Kratzer's algorithm 5.3.18 of [11]. □

Corollary 6.3. *Let* $G = \langle r, s, f, g\rangle$ *be the finitely presented O'Nan group with set* $\mathcal{R}(G)$ *of defining relations given in Theorem* 5.2. *Representatives of the conjugacy classes of the local subgroups* $H = C_G(z)$, $N = N_G(3_A)$, $Y = N_G(5_A)$, $R = N_G(7_A)$ *and the subgroups* $J = \langle r, s, g\rangle$ *and* L *of* G *are given in Tables* A.1, A.5, A.6, A.8, A.4 *and* A.7, *respectively. Using their abbreviations for the conjugacy classes of these subgroups the fusion of the conjugacy classes of these 6 subgroups of* G *into the* 30 *conjugacy classes of* G *is given in Table* A.10.

Proof. The fusion of the conjugacy classes of H and N is given in the previous sections. The fusion of the classes of the other subgroups has been determined computationally by means of MAGMA and the faithful permutation representation of G constructed in Proposition 6.1. □

Corollary 6.4. *Let* $G = \langle r, s, f, g\rangle$ *be the finitely presented group with set* $\mathcal{R}(G)$ *of defining relations given in Theorem* 5.2. *Let* $w = sgr^3gr^7fsfsr^7g^2r^3fgf$ *and* $b = (rs^2)^{3w}$. *Then* L *and* $L_2 = \langle b^{-1}zb, (zb^2)^{-2}b(zb^2)^2\rangle$ *are two subgroups of* G *which are both isomorphic to* $L_3(7) : 2$, *but they are not conjugate in* G.

Proof. As mentioned in the introduction the group $H = C_G(z)$ defined in Lemma 2.1 is isomorphic to the centralizer of that O'Nan group ON for which 2 generating permutations of degree 122.760 are given in Wilson's Atlas [18]. Thus $G \cong \mathsf{ON}$ by Theorem 5.2. By Table A.9 the elements z and $b_1 = (rs^2)^3$ are representatives of the conjugacy classes 2_A and 4_A of G, respectively. Using MAGMA and the faithful permutation representation of G constructed in Proposition 6.1 it has been checked that for the element w given in the statement the elements z and $b = b_1^w$ have a product $p = zb$ of order 11. So z and b^w are standard generators according to Wilson's definition. Now the Atlas [18] provides an explicit recipe to obtain the given generators of L_2 from these standard generators. □

7. Constructing the Character Table

In [12] the first author has given a deterministic algorithm which determines all multiplicity-free irreducible constituents χ_r of a permutation character. Let L and L_2 be the subgroups of the O'Nan group G constructed in Propositions 6.1 and 6.4. In this section we apply this character formula to the permutation characters of the derived subgroups M and M_2 of L and L_2, respectively. Applying then Brauer's character formulas for the 5-blocks of defect 1, we obtain 3 more characters. Using the fusion information on some maximal and some local subgroups, we can induce their characters. We apply the LLL algorithm [10] implemented in GAP 4.4 to the permutation constituents and to the induced characters and obtain the complete character table of the O'Nan group G.

Proposition 7.1. *Let $\mathcal{Z}$ be a commutative semisimple algebra of finite dimension over a separable field F. If $|F|$ is big enough, then $\mathcal{Z} = F[Z]$ for some element $Z \in \mathcal{Z}$. Moreover, we may express the primitive central idempotents of $\mathcal{Z}$ as polynomials in Z over the splitting field E of $\mathcal{Z}$.*

Proof. By Artin–Wedderburn Theorem there exists an F-algebra isomorphism ϕ from $\bigoplus_{i=1}^{r} F_i$ to $\mathcal{Z}$, where F_i/F is finite dimensional. Thus $F_i = F[A_i]$, for some $A_i \in F_i$. Let m_i denote the minimal polynomial of A_i over F. If $|F|$ is big enough, we may modify A_i by non-zero scalars so that the m_i's are pairwise coprime. By the Chinese Remainder Theorem there exist $p_i \in F[x]$ such that $p_i \equiv \delta_{ij} \pmod{m_j}$. Let $Z = \phi(A_1, \ldots, A_r)$, then $\phi(0, \ldots, A_i, \ldots, 0) = Zp_i(Z)$. Since $F_i = F[A_i]$, we get $\mathcal{Z} = F[Z]$. Let $\alpha_{i1}, \ldots, \alpha_{is_i}$ be the roots for the minimal polynomial m_i of A_i in E. Let ℓ_{ij} be the Lagrange polynomial associated to the α_{ij}'s, namely $\ell_{ij}(\alpha_{ik}) = \delta_{jk}$. Then $e_{ij} = \ell_{ij}(Zp_i(Z))$ are the primitive central idempotents of $\mathcal{Z}$. We remark that these polynomials coincide with the Lagrange polynomials associated to all the roots of all m_i's in E. □

Proposition 7.2. *Let $G = \langle r, s, f, g \rangle$ be the finitely presented O'Nan group with set $\mathcal{R}(G)$ of defining relations given in Theorem* 5.2. *Let L and L_2 be the subgroups of G defined in Proposition* 6.1 *and Corollary* 6.4. *Set $M = L'$ and $M_2 = L_2'$. Then the following statements hold*:

(a) *M is the stabilizer of a faithful permutation representation $(1_M)^G$ of G of degree* 245520.
(b) *G is the disjoint union of* 7 *double cosets Mw_iM. The corresponding subdegrees $k_i = |M : M \cap M^{w_i}|$ are* $1, 1, 11172, 12768, 52136, 52136, 117306$.
(c) *The endomorphism ring $\mathcal{E} = \mathrm{End}_{\mathbb{Q}G}((1_M)^G)$ is commutative and isomorphic to the subalgebra $\mathcal{B}$ of $(\mathbb{Q})_7$ generated by $Z = D_4 + D_5$:*

$$D_4 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 432 & 602 & 564 & 564 & 608 \\ 12768 & 12768 & 688 & 698 & 684 & 684 & 640 \\ 0 & 0 & 2632 & 2793 & 2736 & 2736 & 2688 \\ 0 & 0 & 2632 & 2793 & 2736 & 2736 & 2688 \\ 0 & 0 & 6384 & 5880 & 6048 & 6048 & 6144 \end{pmatrix},$$

$$D_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2464 & 2303 & 2394 & 2394 & 2352 \\ 0 & 0 & 2632 & 2793 & 2736 & 2736 & 2688 \\ 52136 & 0 & 11172 & 11172 & 11124 & 10933 & 11088 \\ 0 & 52136 & 11172 & 11172 & 10933 & 11124 & 11088 \\ 0 & 0 & 24696 & 24696 & 24948 & 24948 & 24920 \end{pmatrix}.$$

(d) *Z has eigenvalues* $-184, -152, -72, -51, 343, 383, 64904$.

(e) *The primitive central idempotents of the intersection algebra D are $l_i(Z)$, where the l_i are the Lagrange polynomials corresponding to the eigenvalues of Z.*

(f) *The permutation character $(1_M)^G$ has 7 irreducible constituents χ_i of multiplicity* 1 *having degrees* $1, 10944, 26752, 32395, 37696, 52668, 85064$.

(g) *The same holds for $(1_{M_2})^G$ and all constituents coincide but those of degree* 32395.

(h) *The Gollan Ostermann numbers $m_{j,i} = |\{t \in T : tg_jt^{-1} \in Mw_iM\}|$, where T is a right transversal for M in G, and the values $\chi_r(g_j)$ of the characters obtained above on the* 30 *conjugacy classes of* ON *are reported in Table* 2.

Proof. (a) follows from Proposition 6.1.

(b) MAGMA shows that G has 7 (M, M)-double coset representatives. Furthermore, MAGMA also calculates the 7 subdegrees $k_i = |M : M \cap M^{w_i}|$.

(c) Let $p_{ik}^j = |1^{w_iM} \cap 1^{w_k^{-1}Mw_j}|$, where M is viewed as the stabilizer of G viewed as a permutation group of degree 245520 acting on the M-right cosets. It turns out that the intersection matrices $D_k = (p_{ik}^j)$ generate a commutative algebra $\mathcal{B}$. By Proposition 7.1 we know that $\mathcal{B}$ is a cyclic algebra.

(d) Since all intersection matrices have minimal polynomial of degree at most 6, we looked for generators of shape $D_i + D_j$. MAGMA shows that $Z = D_4 + D_5$ has eigenvalues $-184, -152, -72, -51, 343, 383$, and 64904, hence generates $\mathcal{B}$.

(e) Using Proposition 7.1 we obtain the primitive central idempotents e_r of $\mathcal{B}$ evaluating the Lagrange polynomial ℓ_i relative to the eigenvalues of Z at Z.

(f) We read off the scalars c_{ri} such that $D_ie_r = c_{ri}e_r$. Next we get the Gollan Ostermann numbers:

$$m_{ji} = |\{t \in T : tg_jt^{-1} \in Mw_iM\}|,$$

where T is a right transversal for M in G and g_j^G is the jth conjugacy class of G. Since storing such a transversal would require too much memory we fix $p \in \{1, \dots, 122760\}$ and we construct, using strong generators and Schreier vectors for $G \le S_{122760}$, $g \in G$ such that $p^g = 1$ and browse through all conjugacy classes representatives. Since $\mathcal{B}$ is commutative all irreducible constituents of $(1_M)^G$ occur with multiplicity 1. If χ_r is the unique constituent corresponding to e_r, then it is proved in [12] that

Table 2.

Class	g_j	$\lvert C_G(g_j)\rvert$	$m_{g,1}$	$m_{g,2}$	$m_{g,3}$	$m_{g,4}$	$m_{g,5}$	$m_{g,6}$	$m_{g,7}$	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8
1_a	1	$\lvert \mathsf{ON}\rvert$	245520	0	0	0	0	0	0	10944	26752	32395	32395	37696	52668	85064
2_a	s^8	$2^9\cdot 3^2\cdot 5\cdot 7$	240	480	26880	23040	53760	80640	60480	64	128	75	75	−64	92	−56
3_a	r	$2^3\cdot 3^4\cdot 5$	90	0	11340	12960	68040	51030	102060	9	22	−5	−5	31	18	14
4_a	$(rs^2)^3$	$2^8\cdot 3^2\cdot 5\cdot 7$	0	240	36960	0	40320	67200	100800	64	0	35	35	−64	20	−56
4_b	s^4	2^8	16	0	10976	12544	50304	52736	118944	0	0	3	3	0	4	8
5_a	f	$2^2\cdot 3^2\cdot 5$	0	0	10920	13680	52680	51690	116550	−1	2	0	0	1	−2	−1
6_a	rg	$2^3\cdot 3^2$	6	12	11508	13032	51852	51930	117180	1	2	3	3	−1	2	−2
7_a	$(rs^2f)^2$	$2^2\cdot 7^3$	30	0	19208	10192	57624	47334	111132	17	−2	6	6	15	−7	0
7_b	r^2g	7^2	2	0	12348	11592	52920	52234	116424	3	−2	−1	−1	1	0	0
8_a	s^2	2^5	4	4	11424	13056	51296	51296	118440	0	0	−1	3	0	0	0
8_b	rs	2^5	0	0	11088	12672	52416	52416	116928	0	0	3	−1	0	0	0
10_a	fg	$2^2\cdot 5$	0	0	10640	12280	52260	51270	119070	−1	−2	0	0	1	2	−1
11_a	rgf	11	0	0	10780	12980	51480	52470	117810	−1	0	0	0	−1	0	1
12_a	rs^2	$2^2\cdot 3^2$	0	6	11256	12204	52650	52728	116676	1	0	−1	−1	−1	2	−2
14_a	rs^2f	$2^2\cdot 7$	2	4	11200	12680	52892	53578	115164	1	2	−2	−2	−1	1	0
15_a	fg^2	$3^2\cdot 5$	0	0	10920	13680	52680	51690	116550	−1	2	0	0	1	−2	−1
15_b	$(fg^2)^2$	$3^2\cdot 5$	0	0	10920	13680	52680	51690	116550	−1	2	0	0	1	−2	−1
16_a	s	2^4	2	2	11256	12864	51856	51856	117684	0	0	−1	1	0	0	0
16_b	s^3	2^4	2	2	11256	12864	51856	51856	117684	0	0	−1	1	0	0	0
16_c	r^2s	2^4	0	0	11088	12672	52416	52416	116928	0	0	1	−1	0	0	0
16_d	$(r^2s)^3$	2^4	0	0	11088	12672	52416	52416	116928	0	0	1	−1	0	0	0
19_a	r^2fg	19	2	0	11172	12768	51984	52288	117306	0	0	0	0	0	0	1
19_b	$(r^2fg)^2$	19	2	0	11172	12768	51984	52288	117306	0	0	0	0	0	0	1
19_c	$(r^2fg)^4$	19	2	0	11172	12768	51984	52288	117306	0	0	0	0	0	0	1
20_a	s^2f	$2^2\cdot 5$	0	0	10780	12980	52470	51480	117810	−1	0	0	0	1	0	−1
20_b	$(s^2f)^{11}$	$2^2\cdot 5$	0	0	10780	12980	52470	51480	117810	−1	0	0	0	1	0	−1
28_a	$rsfg$	$2^2\cdot 7$	0	2	11676	12684	51898	52584	116676	1	0	0	0	−1	−1	0
28_b	$(rsfg)^5$	$2^2\cdot 7$	0	2	11676	12684	51898	52584	116676	1	0	0	0	−1	−1	0
31_a	sfg	31	0	0	11718	12462	52080	52080	117180	1	−1	0	0	0	−1	0
31_b	$(sfg)^3$	31	0	0	11718	12462	52080	52080	117180	1	−1	0	0	0	−1	0

(1) $\chi_r(1) = |G : M|(\sum_{k=1}^{7} |M : M \cap M^{w_k}|^{-1} c_{rk}\overline{c_{rk}})^{-1}$.
(2) $\chi_r(g_j) = \frac{\chi_r(1)}{|G|} \sum_{k=1}^{7} c_{rk}|M : M \cap M^{w_k}|m_{jk}$.

(g) We calculate the permutation character of $(1_{M_2})^G$ and reduce it using the constituents of $(1_M)^G$. We are left with an irreducible character of degree 32395.

(h) Using the faithful permutation representation $(1_M)^G$ and Kratzer's algorithm (see [9]), short words in r, s, f, g for the conjugacy classes representatives and their centralizer orders are obtained. In particular the class number of ON is 30.

□

We now apply Brauer's character formula for 5-blocks of defect one.

Proposition 7.3. *The principal* 5*-block* B_0 *of* G *contains* 5 *irreducible characters. They are* $1, \chi_2, \chi_6, \chi_8$ *and*

$$\chi_9 = (58311, 71, -9, 71, 7, 1, -1, 1, 1, -1, -1, 1, 0, -1, 1, 1, 1, -1, -1, -1, -1, 0, 0, 0, 1, 1, 1, 1, 0, 0).$$

Let B *be the* 5*-block containing* χ_3. *Then* B *contains* χ_7 *and two exceptional characters*

$$\chi_{10} = (25916, -36, -4, 20, 4, 1, 0, -5, 2, 0, 0, -1, 0, 2, -1, 1, 1, 0, 0, 0, 0, 0, 0, 0, -i\sqrt{5}, i\sqrt{5}, -1, -1, 0, 0),$$

$$\chi_{11} = (25916, -36, -4, 20, 4, 1, 0, -5, 2, 0, 0, -1, 0, 2, -1, 1, 1, 0, 0, 0, 0, 0, 0, 0, i\sqrt{5}, -i\sqrt{5}, -1, -1, 0, 0).$$

Proof. By [3, Lemma IV.4.2] χ, η are in the same p-block if and only if $\frac{|x^G|\chi(x)}{\chi(1)} \equiv \frac{|x^G|\eta(x)}{\eta(1)} \pmod p$ for all p'-elements x. From Proposition 7.2(h) follows that the set S of irreducibles in $(1_M)^G$ and in B_0 is $\{1, \chi_2, \chi_6, \chi_8\}$. Since $k(b_0)$, the number of characters in the principal 5-block b_0 of $Y = N_G(5_A)$, is 5, there is only one character left in B_0. By Brauer theory $\sum_{\chi \in B_0} \delta_\chi \chi(x) = 0$ for any $5'$-element x, where $\delta_\chi \equiv \chi(1) \pmod 5$. So $\chi_9(x) = -1 + \chi_2(x) - \chi_6(x) + \chi_8(x)$ for any $5'$-element.

Since $\chi_9(1) \equiv 1 \pmod 5$ then $\chi_9(x) = \pm\psi(x)$, where ψ is the Green correspondent of χ_9 in $Y = N_G(5_A)$. Now every ψ in the principal 5-block b_0 of Y assume the same value a on any 5-singular class and $a = \pm 1$. From $[\chi_9, 1_G] = 0$ we get $a = 1$.

Analogously we prove that χ_3 and χ_7 belong to the same 5-block B. As $\chi_3(1) \equiv 2 \pmod 5$ and $\chi_7 \equiv 3 \pmod 5$, [3, Theorem VI.2.14] and Table B.5 of Y imply that the Brauer correspondent b of B in Y has two irreducible characters ψ_3 and ψ_7, say, such that

$$\chi_i(x) = \varepsilon_i \psi_i(x),$$

for all 5-singular elements x and where $\varepsilon_3 = 1 = -\varepsilon_7$.

Another application of [3, Lemma IV.4.2] shows that $k(b) = 4$. We denote ψ_i, $i = 10, 11$ the remaining characters of b. They do not vanish on 5-singular elements. Now [3, Theorem VII.2.15] states that B has two exceptional characters χ_i, $i = 10, 11$, that there is a common sign ε_0 such that $\chi_i(x) = -\varepsilon_0\psi_i(x)$ for $i = 10, 11$, and that $\varepsilon_0\chi_{10}(x) + \chi_3(x) - \chi_7(x) = 0$ for all $5'$-classes, and that

$\chi_{10}(x) = \chi_{11}(x)$ for all 5-regular elements. Notice that $\chi_3(1) + \chi_7(1)$ does not divide $|G|$. Thus $\varepsilon_0 = 1$ and we can obtain χ_i. □

Theorem 7.4. *Let G be the O'Nan group. Then:*

(a) *The character table of G coincides with the one for* ON *given in* [2, *p.* 133].
(b) *G admits a* 495*-dimensional simple module M over $\mathbb{F}_3$.*

Proof. (a) In Corollary 6.2 a complete classification of all conjugacy classes is given. Corollary 6.3 and Proposition 6.1 provide complete information about the local subgroups $N_G(p)$ containing all p-elementary subgroups of G up to conjugation for all prime divisors of $|G|$ which is determined in Theorem 5.2. In particular their conjugacy classes and character tables are given in the Appendix. Using the fusion obtained in Corollary 6.3 we induce the characters of L, L_2, J, H, and $N_G(31_A)$ to G. The calculation of the induced characters was done by means of the following formula $\psi^G(g) = |C_G(g)| \sum_{i=1}^m \frac{\psi(x_i)}{|C_S(x_i)|}$, where $\psi \in \mathrm{Irr}_{\mathbb{C}}(S)$ and $g^S = \bigsqcup_{i=1}^m x_i^S$, for any $S \leq G$, see [7, p. 64]. We add the characters χ_r obtained in Proposition 7.2 to this set. They may be viewed as elements of F^{30}, where F is the cyclotomic field $\mathbb{Q}(\zeta_{164920})$. We construct in GAP 4.4 (see [14]) the integral lattice generated by these vectors with scalar product defined as $[v, w] = \sum_{i=1}^{30} |C_G(g_i)|^{-1} v_i \bar{w}_i$, where the g_i's are the representatives for G-conjugacy classes. An application of the LLL algorithm immediately yields the complete character table of ON.

(b) It is well known that the adjacency matrices A_k belonging to the 5 double cosets Lw_kL form a basis for the endomorphism ring $\mathcal{E} = \mathrm{End}_{\mathbb{C}G}((1_L)^G)$. Applying A_5 to $P = (1_L)^G \otimes \mathbb{F}_3$, one obtains $M = A_5P$. Using MAGMA it has been checked that M is an irreducible $\mathbb{F}_3G$-module of dimension 495. □

Acknowledgments

Both authors thank W. Lempken and M. Weller for helpful group theoretical and computational advice. We also owe sincere thanks to the referee who pointed out several misprints and a serious gap in the previous version of the proof of Proposition 4.2. Furthermore, he suggested an elegant shortcut in the argument of Proposition 3.1.

The second author gratefully acknowledges financial support by the University of Duisburg-Essen and by the Cofin grant for the project "Teoria dei Gruppi e Applicazioni" during his one-month visit at the Institute for Experimental Mathematics in June and July 2003. Both authors thank the Department of Mathematics of Cornell University for its great hospitality. In particular, we owe thanks to K. Dennis for his helpful suggestions and technical support. The second author's visit to Cornell was financially supported by F.A.R. grants of the University of Insubria in Como, Italy.

Appendix A. Representatives of Conjugacy Classes

Table A.1. Conjugacy classes of $H = C_G(z) = \langle c, r, s\rangle$.

i	h_i	$\lvert C_H(h_i)\rvert$	2P	3P	5P	7P
1_a	1	$2^9 \cdot 3^2 \cdot 5 \cdot 7$	1_a	1_a	1_a	1_a
2_a	$(s)^8$	$2^9 \cdot 3^2 \cdot 5 \cdot 7$	1_a	2_a	2_a	2_a
2_b	$(rs)^4$	2^8	1_a	2_b	2_b	2_b
2_c	$(sc^2)^3$	$2^4 \cdot 3^2$	1_a	2_c	2_c	2_c
3_a	r	$2^3 \cdot 3^2$	3_a	1_a	3_a	3_a
4_a	$(rc)^3$	$2^8 \cdot 3^2 \cdot 5 \cdot 7$	2_a	4_a	4_a	4_a
4_b	$(s)^4$	2^8	2_a	4_b	4_b	4_b
4_c	$(rs)^2$	2^7	2_b	4_c	4_c	4_c
4_d	$rcscs$	2^7	2_b	4_d	4_d	4_d
4_e	$rsrsrc$	2^6	2_b	4_e	4_e	4_e
4_f	rcs	2^4	2_b	4_f	4_f	4_f
5_a	c	$2^2 \cdot 5$	5	5	1	5
6_a	$(rc)^2$	$2^3 \cdot 3^2$	3	2_a	6_a	6_a
6_b	sc^2	$2^2 \cdot 3^2$	3	2_c	6_b	6_b
6_c	rsc^2	$2^2 \cdot 3^2$	3	2_c	6_c	6_c
7_a	$(r^2c)^2$	$2^2 \cdot 7$	7_a	7_a	7_a	1_a
8_a	$(s)^2$	2^5	4_b	8_a	8_a	8_a
8_b	$(sc)^2$	2^5	4_b	8_b	8_b	8_b
8_c	rs	2^4	4_c	8_c	8_c	8_c
8_d	rsc	2^4	4_c	8_d	8_d	8_d
10_a	$(s^2c^2)^2$	$2^2 \cdot 5$	5_a	10_a	2_a	10_a
12_a	rc	$2^2 \cdot 3^2$	6_a	4_a	12_a	12_a
14_a	r^2c	$2^2 \cdot 7$	7_a	14_a	14_a	2_a
16_a	s	2^4	8_a	16_b	16_b	16_a
16_b	$(s)^3$	2^4	8_a	16_a	16_a	16_b
16_c	sc	2^4	8_b	16_d	16_d	16_c
16_d	$(sc)^3$	2^4	8_b	16_c	16_c	16_d
20_a	s^2c^2	$2^2 \cdot 5$	10_a	20_a	4_a	20_a
20_b	$(s^2c^2)^{11}$	$2^2 \cdot 5$	10_a	20_b	4_a	20_b
28_a	rc^2	$2^2 \cdot 7$	14_a	28_a	28_b	4_a
28_b	$(rc^2)^5$	$2^2 \cdot 7$	14_a	28_b	28_a	4_a

Table A.2. Conjugacy classes of $D = N_H(A) = \langle r, s\rangle$.

i	d_i	$\lvert C_D(d_i)\rvert$	i	d_i	$\lvert C_D(d_i)\rvert$
1_a	1	$2^9 \cdot 3$	4_g	r^2srs^2	2^4
2_a	$(s)^8$	$2^9 \cdot 3$	6_a	$(rs^2)^2$	$2^2 \cdot 3$
2_b	$(rs)^4$	2^8	8_a	$(s)^2$	2^5
2_c	$rsrs^3$	2^5	8_b	$(r^2s)^2$	2^5
2_d	$r^2s^2rs^3$	2^4	8_c	rs	2^4
3_a	r	$2^2 \cdot 3$	8_d	r^2s^3	2^4
4_a	$(rs^2)^3$	$2^8 \cdot 3$	12_a	rs^2	$2^2 \cdot 3$
4_b	$(s)^4$	2^8	12_b	$(rs^2)^5$	$2^2 \cdot 3$
4_c	$(rs)^2$	2^7	16_a	s	2^4
4_d	$r^2sr^2srs^3rs$	2^7	16_b	$(s)^3$	2^4
4_e	r^2srs^2rs	2^6	16_c	r^2s	2^4
4_f	$r^2sr^2s^3$	2^5	16_d	$(r^2s)^3$	2^4

Table A.3. Conjugacy classes of $E = N_G(A) = \langle r, s, g \rangle$.

i	e_i	$\|C_E(e_i)\|$
1_a	1	$2^9 \cdot 3 \cdot 7$
2_a	$(s)^8$	$2^9 \cdot 3$
2_b	s^2gsg	2^5
3_a	r	$2^2 \cdot 3$
4_a	$(rg)^3$	$2^8 \cdot 3$
4_b	$(s)^4$	2^8
4_c	$rsrg$	2^5
6_a	$(rg)^2$	$2^2 \cdot 3$
7_a	r^2g	7
7_b	$(r^2g)^3$	7
8_a	$(s)^2$	2^5
8_b	$(r^2s)^2$	2^5
12_a	rg	$2^2 \cdot 3$
12_b	$(rg)^5$	$2^2 \cdot 3$
16_a	s	2^4
16_b	$(s)^3$	2^4
16_c	r^2s	2^4
16_d	$(r^2s)^3$	2^4

Table A.4. Conjugacy classes of Janko subgroup $J = \langle z, t, u, f, r, g \rangle$.

i	j_i	$\|C_J(j_i)\|$	2P	3P	5P	7P	11P	19P
1_a	1	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$	1_a	1_a	1_a	1_a	1_a	1_a
2_a	z	$2^3 \cdot 3 \cdot 5$	1_a	2_a	2_a	2_a	2_a	2_a
3_a	r	$2 \cdot 3 \cdot 5$	3_a	1_a	3_a	3_a	3_a	3_a
5_a	f	$2 \cdot 3 \cdot 5$	5_b	5_b	1_a	5_b	5_a	5_a
5_b	$(f)^2$	$2 \cdot 3 \cdot 5$	5_a	5_a	1_a	5_a	5_b	5_b
6_a	zr	$2 \cdot 3$	3_a	2_a	6_a	6_a	6_a	6_a
7_a	r^2g	7	7_a	7_a	7_a	1_a	7_a	7_a
10_a	zf	$2 \cdot 5$	5_b	10_b	2	10_b	10_a	10_a
10_b	$(zf)^3$	$2 \cdot 5$	5_a	10_a	2_a	10_a	10_b	10_b
11_a	frg	11_a	11_a	11_a	11_a	11_a	1_a	11_a
15_a	zfg	$3 \cdot 5$	15_b	5_b	3_a	15_b	15_a	15_a
15_b	$(zfg)^2$	$3 \cdot 5$	15_a	5_a	3_a	15_a	15_b	15_b
19_a	$trgf$	19	19_b	19_b	19_b	19_a	19_a	1_a
19_b	$(trgf)^2$	19	19_c	19_c	19_c	19_b	19_b	1_a
19_c	$(trgf)^4$	19	19_a	19_a	19_a	19_c	19_c	1_a

Table A.5. Conjugacy classes of $N = N_G(3_A) = N_G(r) = \langle r, r_1, n_1, l, w, f_1 \rangle$.

i	n_i	$\lvert C_{N_1}(n_i)\rvert$	2P	3P	5P
1_a	1	$2^4 \cdot 3^4 \cdot 5$	1_a	1_a	1_a
2_a	n_1	$2^4 \cdot 3^2 \cdot 5$	1_a	2_a	2_a
2_b	$(l)^2$	$2^4 \cdot 3^2$	1	2_b	2_b
2_c	$n_1 w$	2^4	1	2_c	2_c
3_a	r	$2^3 \cdot 3^4 \cdot 5$	3_a	1_a	3_a
3_b	r_1	$2^3 \cdot 3^4 \cdot 5$	3_b	1_a	3_b
3_c	rr_1	$2^3 \cdot 3^4 \cdot 5$	3_c	1_a	3_c
3_d	$r^2 r_1$	$2^3 \cdot 3^4 \cdot 5$	3_d	1_a	3_d
3_e	$l f_1$	$2 \cdot 3^4$	3_e	1_a	3_e
3_f	$l^2 f_1$	$2 \cdot 3^4$	3_{f_1}	1	3_{f_1}
3_g	rlf_1	3^4	3_g	1_a	3_g
3_h	$r_1 l f_1$	3^4	3_h	1_a	3_h
3_i	$rr_1 l f_1$	3^4	3_i	1_a	3_i
3_j	$rl^2 f_1$	3^4	3_j	1_a	3_j
3_k	$r_1 l^2 f_1$	3^4	3_k	1_a	3_k
3_l	$r^2 r_1 l f_1$	3^4	3_l	1_a	3_l
3_m	$rr_1 l^2 f_1$	3^4	3_m	1_a	3_m
3_n	$r^2 r_1 l^2 f_1$	3^4	3_n	1_a	3_n
4_a	l	$2^3 \cdot 3^2$	2_b	4_a	4_a
4_b	$n_1 l$	2^3	2_b	4_b	4_b
5_a	f_1	$2 \cdot 3^2 \cdot 5$	5_b	5_b	1
5_b	$(f_1)^2$	$2 \cdot 3^2 \cdot 5$	5_a	5_a	1
6_a	$(rl)^2$	$2^3 \cdot 3^2$	3_a	2_b	6_a
6_b	$(r_1 l)^2$	$2^3 \cdot 3^2$	3_b	2_b	6_b
6_c	$(rr_1 l)^2$	$2^3 \cdot 3^2$	3_c	2_b	6_c
6_d	$(r^2 r_1 l)^2$	$2^3 \cdot 3^2$	3_d	2_b	6_d
6_e	$n_1 l f_1$	$2 \cdot 3^2$	3_e	2_a	6_e
6_{f_1}	$n_1 l^2 f_1$	$2 \cdot 3^2$	3_{f_1}	2_a	6_{f_1}
10_a	$n_1 f_1$	$2 \cdot 5$	5_b	10_b	2_a
10_b	$(n_1 f_1)^3$	$2 \cdot 5$	5_a	10_a	2_a
12_a	rl	$2^2 \cdot 3^2$	6_a	4_a	12_a
12_b	$r_1 l$	$2^2 \cdot 3^2$	6_b	4_a	12_b
12_c	$rr_1 l$	$2^2 \cdot 3^2$	6_c	4_a	12_c
12_d	$r^2 r_1 l$	$2^2 \cdot 3^2$	6_d	4_a	12_d
15_a	rf_1	$3^2 \cdot 5$	15_b	5_b	3_a
15_b	$(rf_1)^2$	$3^2 \cdot 5$	15_a	5_a	3_a
15_c	$r_1 f_1$	$3^2 \cdot 5$	15_d	5_b	3_b
15_d	$(r_1 f_1)^2$	$3^2 \cdot 5$	15_c	5_a	3_b
15_e	$rr_1 f_1$	$3^2 \cdot 5$	15_{f_1}	5_b	3_c
15_f	$(rr_1 f_1)^2$	$3^2 \cdot 5$	15_e	5_a	3_c
15_g	$r^2 r_1 f_1$	$3^2 \cdot 5$	15_h	5_b	3_d
15_h	$(r^2 r_1 f_1)^2$	$3^2 \cdot 5$	15_g	5_a	3_d

Table A.6. Conjugacy classes of $Y = N_G(5_A) = \langle a_1, d_1, q_1 \rangle$.

i	y_i	$\|C_Y(y_i)\|$	2P	3P	5P	i	y_i	$\|C_Y(y_i)\|$	2P	3P	5P
1	1	$2^4 \cdot 3^2 \cdot 5$	1	1	1	4_e	$(a_1q_1)^3$	2^3	2_c	4_d	4_e
2_a	$a_1^2q_1^2$	$2^4 \cdot 3^2$	1	2_a	2_a	4_f	$a_1q_1^2$	2^3	2_b	4_f	4_f
2_b	$(a_1)^{10}$	$2^4 \cdot 5$	1	2_b	2_b	5	$(a_1)^4$	$2^2 \cdot 3^2 \cdot 5$	5	5	1
2_c	$(q_1)^2$	2^4	1	2_c	2_c	6	$a_1^2d_1q_1^2$	$2 \cdot 3^2$	3	2_a	6
3	d_1	$2 \cdot 3^2 \cdot 5$	3	1	3	10	$(a_1)^2$	$2^2 \cdot 5$	5	10	2_b
4_a	$(a_1)^5$	$2^3 \cdot 5$	2_b	4_a	4_a	15_a	$a_1^4d_1$	$3^2 \cdot 5$	15_b	5	3
4_b	q_1	2^3	2_c	4_c	4_b	15_b	$(a_1^4d_1)^2$	$3^2 \cdot 5$	15_a	5	3
4_c	$(q_1)^3$	2^3	2_c	4_b	4_c	20_a	a_1	$2^2 \cdot 5$	10	20_a	4_a
4_d	a_1q_1	2^3	2_c	4_e	4_d	20_b	$(a_1)^{11}$	$2^2 \cdot 5$	10	20_b	4_a

Note: Where the generators a_1, d_1, q_1 have orders 20, 3, and 4, respectively.

Table A.7. Conjugacy classes of $L = \langle p_1, p_2, x_1, n_1 \rangle \simeq L_3(7) : 2$.

i	l_i	$\|C_L(l_i)\|$	2P	3P	7P	19P
1_a	1	$2^6 \cdot 3^2 \cdot 7^3 \cdot 19$	1_a	1_a	1_a	1_a
2_a	$(p_1)^2$	$2^6 \cdot 3 \cdot 7$	1_a	2_a	2_a	2_a
2_b	n_1	$2^5 \cdot 3 \cdot 7$	1_a	2_b	2_b	2_b
3_a	$(p_2)^2$	$2^3 \cdot 3^2$	3_a	1_a	3_a	3_a
4_a	$(p_1n_1)^3$	$2^5 \cdot 3 \cdot 7$	2_a	4_a	4_a	4_a
4_b	p_1	2^5	2_a	4_b	4_b	4_b
6_a	p_2	$2^3 \cdot 3$	3_a	2_a	6_a	6_a
6_b	x_1n_1	$2^2 \cdot 3$	3_a	2_b	6_b	6_b
7_a	$(p_1p_2x_1)^2$	$2^2 \cdot 7^3$	7_a	7_a	1_a	7_a
7_b	p_2x_1	$2 \cdot 7^2$	7_b	7_b	1_a	7_b
7_c	$p_1x_1p_2x_1p_2^2$	7^2	7_c	7_c	1_a	7_c
8_a	$(p_1p_2)^2$	2^5	4_b	8_b	8_a	8_b
8_b	$(p_1p_2)^6$	2^5	4_b	8_a	8_b	8_a
8_c	$p_1p_2n_1p_2$	2^4	4_b	8_c	8_c	8_c
12_a	p_1n_1	$2^2 \cdot 3$	6_a	4_a	12_a	12_a
14_a	$p_1p_2x_1$	$2^2 \cdot 7$	7_a	14_a	2_a	14_a
14_b	$p_1x_1p_1n_1$	$2 \cdot 7$	7_b	14_b	2_b	14_b
16_a	p_1p_2	2^4	8_a	16_b	16_a	16_b
16_b	$(p_1p_2)^3$	2^4	8_b	16_a	16_b	16_a
16_c	p_2n_1	2^4	8_b	16_d	16_c	16_d
16_d	$(p_2n_1)^3$	2^4	8_a	16_c	16_d	16_c
19_a	p_1x_1	19	19_b	19_b	19_a	1_a
19_b	$(p_1x_1)^2$	19	19_c	19_c	19_b	1_a
19_c	$(p_1x_1)^4$	19	19_a	19_a	19_c	1_a
28_a	$p_1x_1n_1$	$2^2 \cdot 7$	14_a	28_a	4_a	28_a
28_b	$(p_1x_1n_1)^5$	$2^2 \cdot 7$	14_a	28_b	4_a	28_b

Table A.8. Conjugacy classes of $N_G(7_A) \simeq R = \langle d, a \rangle$.

i	r_i	$C_R(r_i)$	2P	3P	7P	i	r_i	$C_R(r_i)$	2P	3P	7P
1_a	1	$2^3 \cdot 3 \cdot 7^3$	1_a	1_a	1_a	6_f	$(da)^5$	$2^2 \cdot 3$	3_b	2_c	6_f
2_a	$(d)^{14}$	$2^3 \cdot 3 \cdot 7$	1_a	2_a	2_a	7_a	$(d)^4$	$2^2 \cdot 7^3$	7_a	7_a	1_a
2_b	$(a)^3$	$2^2 \cdot 3 \cdot 7$	1_a	2_b	2_b	7_b	$(da^3)^2$	$2 \cdot 7^2$	7_b	7_b	1_a
2_c	$(da)^3$	$2^2 \cdot 3 \cdot 7$	1_a	2_c	2_c	7_c	$(d^2a^3)^2$	$2 \cdot 7^2$	7_c	7_c	1_a
3_a	$(a)^2$	$2^3 \cdot 3$	3_b	1_a	3_a	7_d	$dada^5$	7^2	7_d	7_d	1_a
3_b	$(a)^4$	$2^3 \cdot 3$	3_a	1_a	3_b	12_a	da^2	$2^2 \cdot 3$	6_a	4_a	12_a
4_a	$(d)^7$	$2^2 \cdot 3 \cdot 7$	2_a	4_a	4_a	12_b	$(da^2)^5$	$2^2 \cdot 3$	6_b	4_a	12_b
6_a	$(da^2)^2$	$2^3 \cdot 3$	3_a	2_a	6_a	14_a	$(d)^2$	$2^2 \cdot 7$	7_a	14_a	2_a
6_b	$(da^2)^{10}$	$2^3 \cdot 3$	3_b	2_a	6_b	14_b	da^3	$2 \cdot 7$	7_b	14_b	2_c
6_c	a	$2^2 \cdot 3$	3_a	2_b	6_c	14_c	d^2a^3	$2 \cdot 7$	7_c	14_c	2_b
6_d	$(a)^5$	$2^2 \cdot 3$	3_b	2_b	6_d	28_a	d	$2^2 \cdot 7$	14_a	28_a	4_a
6_e	da	$2^2 \cdot 3$	3_a	2_c	6_e	28_b	$(d)^5$	$2^2 \cdot 7$	14_a	28_b	4_a

Table A.9. Conjugacy classes of $\mathsf{ON} \simeq G = \langle r, s, f, g \rangle$.

i	g_i	g_i^G	$C_G(g_i)$	2P	3P	5P	7P	11P	19P	31P
1	1	1	$2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$	1	1	1	1	1	1	1
2	s^8	2857239	$2^9 \cdot 3^2 \cdot 5 \cdot 7$	1	2	2	2	2	2	2
3	r	142227008	$2^3 \cdot 3^4 \cdot 5$	3	1	3	3	3	3	3
4_a	$(rs^2)^3$	5714478	$2^8 \cdot 3^2 \cdot 5 \cdot 7$	2	4_a	4_a	4_a	4_a	4_a	4_a
4_b	s^4	1800060570	2^8	2	4_b	4_b	4_b	4_b	4_b	4_b
5	f	2560086144	$2^2 \cdot 3^2 \cdot 5$	5	5	1	5	5	5	5
6	rg	6400215360	$2^3 \cdot 3^2$	3	2	6	6	6	6	6
7_a	$(rs^2f)^2$	335871360	$2^2 \cdot 7^3$	7_a	7_a	7_a	1	7_a	7_a	7_a
7_b	r^2g	9404398080	7^2	7_b	7_b	7_b	1	7_b	7_b	7_b
8_a	s^2	14400484560	2^5	4_b	8_a	8_a	8_a	8_a	8_a	8_a
8_b	rs	14400484560	2^5	4_b	8_b	8_b	8_b	8_b	8_b	8_b
10	fg	23040775296	$2^2 \cdot 5$	5	10	2	10	10	10	10
11	rgf	41892318720	11	11	11	11	11	1	11	11
12	rs^2	12800430720	$2^2 \cdot 3^2$	6	4_a	12	12	12	12	12
14	rs^2f	16457696640	$2^2 \cdot 7$	7_a	14	14	2	14	14	14
15_a	fg^2	10240344576	$3^2 \cdot 5$	15_b	5	3	15_b	15_a	15_a	15_a
15_b	$(fg^2)^2$	10240344576	$3^2 \cdot 5$	15_a	5	3	15_a	15_b	15_b	15_b
16_a	s	28800969120	2^4	8_a	16_b	16_b	16_a	16_b	16_b	16_a
16_b	s^3	28800969120	2^4	8_a	16_a	16_a	16_b	16_a	16_a	16_b
16_c	r^2s	28800969120	2^4	8_b	16_d	16_d	16_c	16_d	16_d	16_c
16_d	$(r^2s)^3$	28800969120	2^4	8_b	16_c	16_c	16_d	16_c	16_c	16_d
19_a	r^2fg	24253447680	19	19_b	19_b	19_b	19_a	19_a	1	19_a
19_b	$(r^2fg)^2$	24253447680	19	19_c	19_c	19_c	19_b	19_b	1	19_b
19_c	$(r^2fg)^4$	24253447680	19	19_a	19_a	19_a	19_c	19_c	1	19_c
20_a	s^2f	23040775296	$2^2 \cdot 5$	10	20_a	4_a	20_a	20_b	20_b	20_b
20_b	$(s^2f)^{11}$	23040775296	$2^2 \cdot 5$	10	20_b	4_a	20_b	20_a	20_a	20_a
28_a	$rsfg$	16457696640	$2^2 \cdot 7$	14	28_a	28_b	4_a	28_b	28_a	28_a
28_b	$(rsfg)^5$	16457696640	$2^2 \cdot 7$	14	28_b	28_a	4_a	28_a	28_b	28_b
31_a	sfg	14865016320	31	31_a	31_b	31_a	31_a	31_b	31_a	1
31_b	$(sfg)^3$	14865016320	31	31_b	31_a	31_b	31_b	31_a	31_b	1

Table A.10. Fusion table of conjugacy classes of seven subgroups in G.

x	$\|C_G(x)\|$	$C_G(2_A)$	$N_G(3_A)$	$N_G(5_A)$	$N_G(7_A)$	J	L
1_A	$\|G\|$	a	a	a	a	a	a
2_A	$2^9 \cdot 3^2 \cdot 5 \cdot 7$	a, b, c	a, b, c	a, b, c	a, b, c	a	a, b
3_A	$2^3 \cdot 3^4 \cdot 5$	a	$a, \ldots, n$	a	a, b	a	a
4_A	$2^8 \cdot 3^2 \cdot 5 \cdot 7$	a, d	a	a, b	a		a
4_B	2^8	b, c, e, f	b	c, d, e, f			b
5_A	$2^2 \cdot 3^2 \cdot 5$	a	a, b	a		a, b	
6_A	$2^3 \cdot 3^2$	a, b, c	$a, \ldots, f$	a	$a, \ldots, f$		a, b
7_A	$2^2 \cdot 7^3$	a			a, b, c	a	a, b
7_B	7^2				d		c
8_A	2^5	a, d					a, b, c
8_B	2^5	b, c					
10_A	$2^5 \cdot 5$	a	a, b	a		a, b	
11_A	11					a	
12_A	$2^2 \cdot 3^2$	a	a, b, c, d		a, b		a
14_A	$2^2 \cdot 7$	a			a, b, c		a, b
15_A	$3^2 \cdot 5$		a, c, e, g	a		a	
15_B	$3^2 \cdot 5$		b, d, f, h	b		b	
16_A	2^4	a					a, c
16_B	2^4	b					b, d
16_C	2^4	c					
16_D	2^4	d					
19_A	19	a				a	b
19_B	19	b				b	c
19_C	19	c				c	a
20_A	$2^2 \cdot 5$	a		a			
20_B	$2^2 \cdot 5$	b		b			
28_A	$2^2 \cdot 7$	a			a, b		b
28_B	$2^2 \cdot 7$	b					a
31_A	31						
31_B	31						

Appendix B. Character Tables

Table B.1. Character table of $H = C_G(z)$.

	1a	2a	2b	2c	3a	4a	4b	4c	4d	4e	4f	5a	6a	6b	6c	7a	8a	8b	8c	8d	10a	12a	14a	16a	16b	16c	16d	20a	20b	28a	28b
2	9	9	8	4	3	8	8	7	7	6	4	2	3	2	2	2	5	5	4	4	2	2	2	4	4	4	4	2	2	2	2
3	2	2	.	2	2	2	.	.	.	.	.	.	2	2	2	.	.	.	.	.	.	2	.	.	.	.	.	.	.	.	.
5	1	1	.	.	.	1	.	.	.	.	.	1	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	1	1	.	.
7	1	1	.	.	.	1	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	1	.	.	.	.	.	.	1	1
2P	1a	1a	1a	1a	3a	2a	2a	2b	2b	2b	2b	5a	3a	3a	3a	7a	4b	4b	4d	4d	5a	6a	7a	8a	8b	8a	8b	10a	10a	14a	14a
3P	1a	2a	2b	2c	1a	4a	4b	4c	4d	4e	4f	5a	2a	2c	2c	7a	8a	8b	8c	8d	10a	4a	14a	16c	16d	16a	16b	20a	20b	28a	28b
5P	1a	2a	2b	2c	3a	4a	4b	4c	4d	4e	4f	1a	6a	6b	6c	7a	8a	8b	8c	8d	2a	12a	14a	16c	16d	16a	16b	4a	4a	28b	28a
7P	1a	2a	2b	2c	3a	4a	4b	4c	4d	4e	4f	5a	6a	6b	6c	1a	8a	8b	8c	8d	10a	12a	2a	16a	16b	16c	16d	20a	20b	4a	4a
11P	1a	2a	2b	2c	3a	4a	4b	4c	4d	4e	4f	5a	6a	6b	6c	7a	8a	8b	8c	8d	10a	12a	14a	16c	16d	16a	16b	20b	20a	28b	28a
13P	1a	2a	2b	2c	3a	4a	4b	4c	4d	4e	4f	5a	6a	6b	6c	7a	8a	8b	8c	8d	10a	12a	14a	16c	16d	16a	16b	20b	20a	28b	28a
17P	1a	2a	2b	2c	3a	4a	4b	4c	4d	4e	4f	5a	6a	6b	6c	7a	8a	8b	8c	8d	10a	12a	14a	16a	16b	16c	16d	20b	20a	28b	28a
19P	1a	2a	2b	2c	3a	4a	4b	4c	4d	4e	4f	5a	6a	6b	6c	7a	8a	8b	8c	8d	10a	12a	14a	16c	16d	16a	16b	20b	20a	28a	28b
23P	1a	2a	2b	2c	3a	4a	4b	4c	4d	4e	4f	5a	6a	6b	6c	7a	8a	8b	8c	8d	10a	12a	14a	16a	16b	16c	16d	20a	20b	28b	28a
X.1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.2	1	1	1	−1	1	1	1	1	1	1	−1	1	1	−1	−1	1	1	1	−1	−1	1	1	1	−1	−1	−1	−1	1	1	1	1
X.3	20	20	4	.	2	−20	−4	4	4	−4	.	.	2	.	.	−1	.	.	.	.	.	−2	−1	.	.	.	.	.	.	1	1
X.4	20	20	4	−2	2	20	4	.	.	.	2	.	2	−2	−2	−1	.	.	.	.	.	2	−1	.	.	.	.	.	.	−1	−1
X.5	20	20	4	2	2	20	4	.	.	.	−2	.	2	2	2	−1	.	.	.	.	.	2	−1	.	.	.	.	.	.	−1	−1
X.6	35	35	3	−1	−1	35	3	3	3	3	−1	.	−1	−1	−1	.	−1	−1	−1	−1	.	−1	.	1	1	1	1	.	.	.	.
X.7	35	35	3	−1	−1	35	3	−1	−1	−1	−1	.	−1	−1	−1	.	3	−1	1	1	.	−1	.	−1	1	−1	1	.	.	.	.
X.8	35	35	3	−1	−1	35	3	−1	−1	−1	−1	.	−1	−1	−1	.	−1	3	1	1	.	−1	.	1	−1	1	−1	.	.	.	.
X.9	35	35	3	1	−1	35	3	3	3	3	1	.	−1	1	1	.	−1	−1	1	1	.	−1	.	−1	−1	−1	−1	.	.	.	.
X.10	35	35	3	1	−1	35	3	−1	−1	−1	1	.	−1	1	1	.	3	−1	−1	−1	.	−1	.	1	−1	1	−1	.	.	.	.
X.11	35	35	3	1	−1	35	3	−1	−1	−1	1	.	−1	1	1	.	−1	3	−1	−1	.	−1	.	−1	1	−1	1	.	.	.	.
X.12	36	36	4	.	.	−36	−4	.	.	.	.	1	.	.	.	1	.	.	2	−2	1	.	1	.	.	.	.	−1	−1	−1	−1
X.13	36	36	4	.	.	−36	−4	.	.	.	.	1	.	.	.	1	.	.	−2	2	1	.	1	.	.	.	.	−1	−1	−1	−1
X.14	40	−40	.	.	4	.	.	−4	4	.	.	.	−4	.	.	−2	.	.	.	.	.	.	2	.	.	.	.	.	.	.	.
X.15	56	56	−8	.	2	−56	8	.	.	.	.	1	2	.	.	.	.	.	.	.	1	−2	.	.	.	.	.	−1	−1	.	.
X.16	56	−56	.	.	2	.	.	4	−4	.	.	1	−2	.	.	.	.	.	.	.	−1	.	.	.	.	.	.	B	−B	.	.
X.17	56	−56	.	.	2	.	.	4	−4	.	.	1	−2	.	.	.	.	.	.	.	−1	.	.	.	.	.	.	−B	B	.	.
X.18	64	64	.	.	1	−64	.	.	.	.	.	−1	1	3	−3	1	.	.	.	.	−1	−1	1	.	.	.	.	1	1	−1	−1
X.19	64	64	.	.	1	−64	.	.	.	.	.	−1	1	−3	3	1	.	.	.	.	−1	−1	1	.	.	.	.	1	1	−1	−1
X.20	64	64	.	8	1	64	.	.	.	.	.	−1	1	−1	−1	1	.	.	.	.	−1	1	1	.	.	.	.	−1	−1	1	1
X.21	64	64	.	−8	1	64	.	.	.	.	.	−1	1	1	1	1	.	.	.	.	−1	1	1	.	.	.	.	−1	−1	1	1
X.22	70	70	−2	.	−2	−70	2	2	2	−2	.	.	−2	.	.	.	.	.	.	.	.	2	.	A	−A	−A	A	.	.	.	.
X.23	70	70	−2	.	−2	−70	2	2	2	−2	.	.	−2	.	.	.	.	.	.	.	.	2	.	−A	A	A	−A	.	.	.	.
X.24	72	−72	.	.	.	.	.	−4	4	.	.	2	.	.	.	2	.	.	.	.	−2	.	−2	.	.	.	.	.	.	.	.
X.25	90	90	2	.	.	−90	−2	−2	−2	2	.	.	.	.	.	−1	.	.	.	.	.	.	−1	A	A	−A	−A	.	.	1	1
X.26	90	90	2	.	.	−90	−2	−2	−2	2	.	.	.	.	.	−1	.	.	.	.	.	.	−1	−A	−A	A	A	.	.	1	1
X.27	90	90	−6	.	.	90	−6	2	2	2	.	.	.	.	.	−1	2	2	.	.	.	.	−1	.	.	.	.	.	.	−1	−1
X.28	126	126	−2	.	.	126	−2	−2	−2	−2	.	1	.	.	.	.	−2	−2	.	.	1	.	.	.	.	.	.	1	1	.	.
X.29	128	−128	.	.	2	.	.	.	.	.	.	−2	−2	.	.	2	.	.	.	.	2	.	−2	.	.	.	.	.	.	.	.
X.30	160	−160	.	.	−2	.	.	.	.	.	.	.	2	.	.	−1	.	.	.	.	.	.	1	.	.	.	.	.	.	C	−C
X.31	160	−160	.	.	−2	.	.	.	.	.	.	.	2	.	.	−1	.	.	.	.	.	.	1	.	.	.	.	.	.	−C	C

Note: Where $A = \sqrt{2}$, $B = i\sqrt{5}$ and $C = \sqrt{7}$.

Table B.2. Character table of $D = N_H(A)$.

2	9	9	8	5	4	2	8	8	7	7	6	5	4	2	5	5	4	4	2	2	4	4	4	4
3	1	1	.	.	.	1	1	.	.	.	.	.	.	1	.	.	.	.	1	1	.	.	.	.
	$1a$	$2a$	$2b$	$2c$	$2d$	$3a$	$4a$	$4b$	$4c$	$4d$	$4e$	$4f$	$4g$	$6a$	$8a$	$8b$	$8c$	$8d$	$12a$	$12b$	$16a$	$16b$	$16c$	$16d$
$2P$	$1a$	$1a$	$1a$	$1a$	$1a$	$3a$	$2a$	$2a$	$2b$	$2b$	$2b$	$2a$	$2b$	$3a$	$4b$	$4b$	$4d$	$4d$	$6a$	$6a$	$8a$	$8a$	$8b$	$8b$
$3P$	$1a$	$2a$	$2b$	$2c$	$2d$	$1a$	$4a$	$4b$	$4c$	$4d$	$4e$	$4f$	$4g$	$2a$	$8a$	$8b$	$8c$	$8d$	$4a$	$4a$	$16b$	$16a$	$16d$	$16c$
$X.1$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$X.2$	1	1	1	1	−1	1	1	1	1	1	1	1	−1	1	1	1	−1	−1	1	1	−1	−1	−1	−1
$X.3$	2	2	2	2	.	−1	2	2	2	2	2	2	.	−1	2	2	.	.	−1	−1	.	.	.	.
$X.4$	3	3	3	−1	−1	.	3	3	3	3	3	−1	−1	.	−1	−1	−1	−1	.	.	1	1	1	1
$X.5$	3	3	3	−1	1	.	3	3	3	3	3	−1	1	.	−1	−1	1	1	.	.	−1	−1	−1	−1
$X.6$	3	3	3	−1	−1	.	3	3	−1	−1	−1	−1	−1	.	−1	3	1	1	.	.	1	1	−1	−1
$X.7$	3	3	3	−1	1	.	3	3	−1	−1	−1	−1	1	.	−1	3	−1	−1	.	.	−1	−1	1	1
$X.8$	3	3	3	−1	−1	.	3	3	−1	−1	−1	−1	−1	.	3	−1	1	1	.	.	−1	−1	1	1
$X.9$	3	3	3	−1	1	.	3	3	−1	−1	−1	−1	1	.	3	−1	−1	−1	.	.	1	1	−1	−1
$X.10$	4	4	4	.	.	1	−4	−4	.	.	.	.	.	1	.	.	−2	2	−1	−1	.	.	.	.
$X.11$	4	4	4	.	.	1	−4	−4	.	.	.	.	.	1	.	.	2	−2	−1	−1	.	.	.	.
$X.12$	6	6	6	2	.	.	6	6	−2	−2	−2	2	.	.	−2	−2	.	.	.	.	.	.	.	.
$X.13$	6	6	−2	−2	.	.	−6	2	2	2	−2	2	.	.	.	.	.	.	.	.	B	$-B$	$-B$	B
$X.14$	6	6	−2	−2	.	.	−6	2	2	2	−2	2	.	.	.	.	.	.	.	.	$-B$	B	B	$-B$
$X.15$	6	6	−2	2	.	.	−6	2	2	2	−2	−2	.	.	.	.	.	.	.	.	B	$-B$	B	$-B$
$X.16$	6	6	−2	2	.	.	−6	2	2	2	−2	−2	.	.	.	.	.	.	.	.	$-B$	B	$-B$	B
$X.17$	8	−8	.	.	.	2	.	.	−4	4	.	.	.	−2	.	.	.	.	.	.	.	.	.	.
$X.18$	8	−8	.	.	.	−1	.	.	−4	4	.	.	.	1	.	.	.	.	A	$-A$	.	.	.	.
$X.19$	8	−8	.	.	.	−1	.	.	−4	4	.	.	.	1	.	.	.	.	$-A$	A	.	.	.	.
$X.20$	8	8	8	.	.	−1	−8	−8	.	.	.	.	.	−1	.	.	.	.	1	1	.	.	.	.
$X.21$	12	12	−4	.	.	.	−12	4	−4	−4	4	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.22$	12	12	−4	.	2	.	12	−4	.	.	.	.	−2	.	.	.	.	.	.	.	.	.	.	.
$X.23$	12	12	−4	.	−2	.	12	−4	.	.	.	.	2	.	.	.	.	.	.	.	.	.	.	.
$X.24$	24	−24	.	.	.	.	.	.	4	−4	.	.	.	.	.	.	.	.	.	.	.	.	.	.

Note: Where $A = -\sqrt{3}$ and $B = -\sqrt{2}$.

Table B.3. Character table of $E = N_G(A)$.

2	9	9	5	2	8	8	5	2	.	.	5	5	2	2	4	4	4	4
3	1	1	.	1	1	.	.	1	.	.	.	.	1	1	.	.	.	.
7	1	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.
	1a	2a	2b	3a	4a	4b	4c	6a	7a	7b	8a	8b	12a	12b	16a	16b	16c	16d
2P	1a	1a	1a	3a	2a	2a	2a	3a	7a	7b	4b	4b	6a	6a	8a	8a	8b	8b
3P	1a	2a	2b	1a	4a	4b	4c	2a	7b	7a	8a	8b	4a	4a	16b	16a	16d	16c
5P	1a	2a	2b	3a	4a	4b	4c	6a	7b	7a	8a	8b	12b	12a	16b	16a	16d	16c
7P	1a	2a	2b	3a	4a	4b	4c	6a	1a	1a	8a	8b	12b	12a	16a	16b	16c	16d
11P	1a	2a	2b	3a	4a	4b	4c	6a	7a	7b	8a	8b	12a	12b	16b	16a	16d	16c
13P	1a	2a	2b	3a	4a	4b	4c	6a	7b	7a	8a	8b	12a	12b	16b	16a	16d	16c
X.1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.2	3	3	−1	.	3	3	−1	.	A	/A	−1	−1	.	.	1	1	1	1
X.3	3	3	−1	.	3	3	−1	.	/A	A	−1	−1	.	.	1	1	1	1
X.4	6	6	2	.	6	6	2	.	−1	−1	2	2	.	.	.	.	.	.
X.5	7	7	−1	1	7	7	−1	1	.	.	−1	−1	1	1	−1	−1	−1	−1
X.6	7	7	−1	1	−1	−1	−1	1	.	.	−1	3	−1	−1	1	1	−1	−1
X.7	7	7	−1	1	−1	−1	−1	1	.	.	3	−1	−1	−1	−1	−1	1	1
X.8	8	8	.	−1	8	8	.	−1	1	1	.	.	−1	−1	.	.	.	.
X.9	14	14	−2	−1	−2	−2	−2	−1	.	.	2	2	1	1	.	.	.	.
X.10	21	21	1	.	−3	−3	1	.	.	.	−3	1	.	.	−1	−1	1	1
X.11	21	21	1	.	−3	−3	1	.	.	.	1	−3	.	.	1	1	−1	−1
X.12	28	−4	.	−2	−12	4	.	2	.	.	.	.	.	.	.	.	.	.
X.13	28	−4	.	1	−12	4	.	−1	.	.	.	.	B	−B	.	.	.	.
X.14	28	−4	.	1	−12	4	.	−1	.	.	.	.	−B	B	.	.	.	.
X.15	42	−6	−2	.	6	−2	2	.	.	.	.	.	.	.	C	−C	C	−C
X.16	42	−6	−2	.	6	−2	2	.	.	.	.	.	.	.	−C	C	−C	C
X.17	42	−6	2	.	6	−2	−2	.	.	.	.	.	.	.	C	−C	−C	C
X.18	42	−6	2	.	6	−2	−2	.	.	.	.	.	.	.	−C	C	C	−C

Note: Where $A = \frac{1}{2}(-1 + i\sqrt{7})$, $B = -\sqrt{3}$ and $C = \sqrt{2}$.

Table B.4. Character table of $N_1 = N_G(r)$.

2	4	4	4	4	3	3	3	3	1	1	.	.	.	.	.	.	.	.	3	3	.	3
3	4	2	1	1	4	4	4	4	4	4	4	4	4	4	4	4	4	4	2	.	2	2
5	1	.	.	.	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	1	.
	1a	2a	2b	2c	3a	3b	3c	3d	3e	3f	3g	3h	3i	3j	3k	3l	3m	3n	4a	4b	5a	6a
2P	1a	1a	1a	1a	3a	3b	3c	3d	3e	3f	3g	3h	3i	3j	3k	3l	3m	3n	2a	2a	5a	3b
3P	1a	2a	2b	2c	1a	1a	1a	1a	1a	1a	1a	1a	1a	1a	1a	1a	1a	1a	4a	4b	5a	2a
5P	1a	2a	2b	2c	3a	3b	3c	3d	3e	3f	3g	3h	3i	3j	3k	3l	3m	3n	4a	4b	1a	6a
X.1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.2	1	1	−1	−1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	−1	1	1
X.3	2	2	.	.	−1	−1	−1	2	2	2	2	−1	−1	2	−1	−1	−1	−1	2	.	2	−1
X.4	2	2	.	.	−1	−1	2	−1	2	2	−1	2	−1	−1	−1	2	−1	−1	2	.	2	−1
X.5	2	2	.	.	2	−1	−1	−1	2	2	−1	−1	−1	−1	2	−1	−1	2	2	.	2	−1
X.6	2	2	.	.	−1	2	−1	−1	2	2	−1	−1	2	−1	−1	−1	2	−1	2	.	2	2
X.7	5	1	3	−1	5	5	5	5	−1	2	2	2	−1	−1	−1	−1	2	2	−1	1	.	1
X.8	5	1	−3	1	5	5	5	5	−1	2	2	2	−1	−1	−1	−1	2	2	−1	−1	.	1
X.9	5	1	1	−3	5	5	5	5	2	−1	−1	−1	2	2	2	2	−1	−1	−1	−1	.	1
X.10	5	1	−1	3	5	5	5	5	2	−1	−1	−1	2	2	2	2	−1	−1	−1	1	.	1
X.11	9	1	3	3	9	9	9	9	.	.	.	.	.	.	.	.	.	.	1	−1	−1	1
X.12	9	1	−3	−3	9	9	9	9	.	.	.	.	.	.	.	.	.	.	1	1	−1	1
X.13	10	2	.	.	−5	−5	−5	10	−2	4	4	−2	1	−2	1	1	−2	−2	−2	.	.	−1
X.14	10	2	.	.	−5	−5	−5	10	4	−2	−2	1	−2	4	−2	−2	1	1	−2	.	.	−1
X.15	10	−2	2	−2	10	10	10	10	1	1	1	1	1	1	1	1	1	1	.	.	.	−2
X.16	10	−2	−2	2	10	10	10	10	1	1	1	1	1	1	1	1	1	1	.	.	.	−2
X.17	10	2	.	.	−5	−5	10	−5	−2	4	−2	4	1	1	1	−2	−2	−2	−2	.	.	−1
X.18	10	2	.	.	−5	−5	10	−5	4	−2	1	−2	−2	−2	−2	4	1	1	−2	.	.	−1
X.19	10	2	.	.	10	−5	−5	−5	−2	4	−2	−2	1	1	−2	1	−2	4	−2	.	.	−1
X.20	10	2	.	.	10	−5	−5	−5	4	−2	1	1	−2	−2	4	−2	1	−2	−2	.	.	−1
X.21	10	2	.	.	−5	10	−5	−5	−2	4	−2	−2	−2	1	1	1	4	−2	−2	.	.	2
X.22	10	2	.	.	−5	10	−5	−5	4	−2	1	1	4	−2	−2	−2	−2	1	−2	.	.	2
X.23	16	.	.	.	−8	−8	−8	16	−2	−2	−2	1	1	−2	1	1	1	1	.	.	1	.
X.24	16	.	.	.	−8	−8	−8	16	−2	−2	−2	1	1	−2	1	1	1	1	.	.	1	.
X.25	16	.	.	.	16	16	16	16	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	.	.	1	.
X.26	16	.	.	.	−8	−8	16	−8	−2	−2	1	−2	1	1	1	−2	1	1	.	.	1	.
X.27	16	.	.	.	−8	−8	16	−8	−2	−2	1	−2	1	1	1	−2	1	1	.	.	1	.
X.28	16	.	.	.	16	−8	−8	−8	−2	−2	1	1	1	1	−2	1	1	−2	.	.	1	.
X.29	16	.	.	.	16	−8	−8	−8	−2	−2	1	1	1	1	−2	1	1	−2	.	.	1	.
X.30	16	.	.	.	−8	16	−8	−8	−2	−2	1	1	−2	1	1	1	−2	1	.	.	1	.
X.31	16	.	.	.	−8	16	−8	−8	−2	−2	1	1	−2	1	1	1	−2	1	.	.	1	.
X.32	18	2	.	.	−9	−9	−9	18	.	.	.	.	.	.	.	.	.	.	2	.	−2	−1
X.33	18	2	.	.	−9	−9	18	−9	.	.	.	.	.	.	.	.	.	.	2	.	−2	−1
X.34	18	2	.	.	18	−9	−9	−9	.	.	.	.	.	.	.	.	.	.	2	.	−2	−1
X.35	18	2	.	.	−9	18	−9	−9	.	.	.	.	.	.	.	.	.	.	2	.	−2	2
X.36	20	−4	.	.	−10	−10	−10	20	2	2	2	−1	−1	2	−1	−1	−1	−1	.	.	.	2
X.37	20	−4	.	.	−10	−10	20	−10	2	2	−1	2	−1	−1	−1	2	−1	−1	.	.	.	2
X.38	20	−4	.	.	20	−10	−10	−10	2	2	−1	−1	−1	−1	2	−1	−1	2	.	.	.	2
X.39	20	−4	.	.	−10	20	−10	−10	2	2	−1	−1	2	−1	−1	−1	2	−1	.	.	.	−4

Table B.4. (*Continued*)

2	3	3	3	1	1	2	2	2	2	.	.	.	.	.	.	.	.
3	2	2	2	1	1	2	2	2	2	2	2	2	2	2	2	2	2
5	.	.	.	.	.	.	.	.	.	1	1	1	1	1	1	1	1
	$6b$	$6c$	$6d$	$6e$	$6f$	$12a$	$12b$	$12c$	$12d$	$15a$	$15b$	$15c$	$15d$	$15e$	$15f$	$15g$	$15h$
$2P$	$3a$	$3d$	$3c$	$3f$	$3e$	$6d$	$6c$	$6b$	$6a$	$15a$	$15b$	$15c$	$15d$	$15e$	$15f$	$15g$	$15h$
$3P$	$2a$	$2a$	$2a$	$2b$	$2c$	$4a$	$4a$	$4a$	$4a$	$5a$	$5a$	$5a$	$5a$	$5a$	$5a$	$5a$	$5a$
$5P$	$6b$	$6c$	$6d$	$6e$	$6f$	$12a$	$12b$	$12c$	$12d$	$3d$	$3a$	$3c$	$3b$	$3c$	$3a$	$3b$	$3d$
$X.1$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$X.2$	1	1	1	−1	−1	1	1	1	1	1	1	1	1	1	1	1	1
$X.3$	−1	2	−1	.	.	−1	2	−1	−1	2	−1	−1	−1	−1	−1	−1	2
$X.4$	−1	−1	2	.	.	2	−1	−1	−1	−1	−1	2	−1	2	−1	−1	−1
$X.5$	2	−1	−1	.	.	−1	−1	2	−1	−1	2	−1	−1	−1	2	−1	−1
$X.6$	−1	−1	−1	.	.	−1	−1	−1	2	−1	−1	−1	2	−1	−1	2	−1
$X.7$	1	1	1	.	−1	−1	−1	−1	−1	.	.	.	.	.	.	.	.
$X.8$	1	1	1	.	1	−1	−1	−1	−1	.	.	.	.	.	.	.	.
$X.9$	1	1	1	1	.	−1	−1	−1	−1	.	.	.	.	.	.	.	.
$X.10$	1	1	1	−1	.	−1	−1	−1	−1	.	.	.	.	.	.	.	.
$X.11$	1	1	1	.	.	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1
$X.12$	1	1	1	.	.	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1
$X.13$	−1	2	−1	.	.	1	−2	1	1	.	.	.	.	.	.	.	.
$X.14$	−1	2	−1	.	.	1	−2	1	1	.	.	.	.	.	.	.	.
$X.15$	−2	−2	−2	−1	1	.	.	.	.	.	.	.	.	.	.	.	.
$X.16$	−2	−2	−2	1	−1	.	.	.	.	.	.	.	.	.	.	.	.
$X.17$	−1	−1	2	.	.	−2	1	1	1	.	.	.	.	.	.	.	.
$X.18$	−1	−1	2	.	.	−2	1	1	1	.	.	.	.	.	.	.	.
$X.19$	2	−1	−1	.	.	1	1	−2	1	.	.	.	.	.	.	.	.
$X.20$	2	−1	−1	.	.	1	1	−2	1	.	.	.	.	.	.	.	.
$X.21$	−1	−1	−1	.	.	1	1	1	−2	.	.	.	.	.	.	.	.
$X.22$	−1	−1	−1	.	.	1	1	1	−2	.	.	.	.	.	.	.	.
$X.23$	.	.	.	.	.	.	.	.	.	1	A	$/A$	A	A	$/A$	$/A$	1
$X.24$	.	.	.	.	.	.	.	.	.	1	$/A$	A	$/A$	$/A$	A	A	1
$X.25$	.	.	.	.	.	.	.	.	.	1	1	1	1	1	1	1	1
$X.26$	.	.	.	.	.	.	.	.	.	A	A	1	$/A$	1	$/A$	A	$/A$
$X.27$	.	.	.	.	.	.	.	.	.	$/A$	$/A$	1	A	1	A	$/A$	A
$X.28$	.	.	.	.	.	.	.	.	.	A	1	A	A	$/A$	1	$/A$	$/A$
$X.29$	.	.	.	.	.	.	.	.	.	$/A$	1	$/A$	$/A$	A	1	A	A
$X.30$	.	.	.	.	.	.	.	.	.	A	$/A$	$/A$	1	A	A	1	$/A$
$X.31$	.	.	.	.	.	.	.	.	.	$/A$	A	A	1	$/A$	$/A$	1	A
$X.32$	−1	2	−1	.	.	−1	2	−1	−1	−2	1	1	1	1	1	1	−2
$X.33$	−1	−1	2	.	.	2	−1	−1	−1	1	1	−2	1	−2	1	1	1
$X.34$	2	−1	−1	.	.	−1	−1	2	−1	1	−2	1	1	1	−2	1	1
$X.35$	−1	−1	−1	.	.	−1	−1	−1	2	1	1	1	−2	1	1	−2	1
$X.36$	2	−4	2	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.37$	2	2	−4	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.38$	−4	2	2	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.39$	2	2	2	.	.	.	.	.	.	.	.	.	.	.	.	.	.

Note: Where $A = -\frac{1}{2}(1 + i\sqrt{15})$.

Table B.5. Character table of $Y = N_G(f) = N_G(5A)$.

2	4	4	4	4	1	3	3	3	3	3	3	2	1	2	.	.	2	2
3	2	2	.	.	2	.	.	.	.	.	.	2	2	.	2	2	.	.
5	1	.	1	.	1	1	.	.	.	.	.	1	.	1	1	1	1	1
	1_a	2_a	2_b	2_c	3_a	4_a	4_b	4_c	4_d	4_e	4_f	5_a	6_a	10_a	15_a	15_b	20_a	20_b
$2P$	1_a	1_a	1_a	1_a	3_a	2_b	2_c	2_b	2_c	2_c	2_c	5_a	3_a	5_a	15_b	15_a	10_a	10_a
$3P$	1_a	2_a	2_b	2_c	1_a	4_a	4_d	4_c	4_b	4_f	4_e	5_a	2_a	10_a	5_a	5_a	20_a	20_b
$5P$	1_a	2_a	2_b	2_c	3_a	4_a	4_b	4_c	4_d	4_e	4_f	1_a	6_a	2_b	3_a	3_a	4_a	4_a
$X.1$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$X.2$	1	1	1	1	1	−1	1	−1	1	−1	−1	1	1	1	1	1	−1	−1
$X.3$	1	−1	1	−1	1	1	$-A$	−1	A	A	$-A$	1	−1	1	1	1	1	1
$X.4$	1	−1	1	−1	1	−1	$-A$	1	A	$-A$	A	1	−1	1	1	1	−1	−1
$X.5$	1	1	1	1	1	1	−1	1	−1	−1	−1	1	1	1	1	1	1	1
$X.6$	1	1	1	1	1	−1	−1	−1	−1	1	1	1	1	1	1	1	−1	−1
$X.7$	1	−1	1	−1	1	1	A	−1	$-A$	$-A$	A	1	−1	1	1	1	1	1
$X.8$	1	−1	1	−1	1	−1	A	1	$-A$	A	$-A$	1	−1	1	1	1	−1	−1
$X.9$	2	2	−2	−2	2	.	.	.	.	.	.	2	2	−2	2	2	.	.
$X.10$	2	−2	−2	2	2	.	.	.	.	.	.	2	−2	−2	2	2	.	.
$X.11$	4	.	4	.	4	4	.	.	.	.	.	−1	.	−1	−1	−1	−1	−1
$X.12$	4	.	4	.	4	−4	.	.	.	.	.	−1	.	−1	−1	−1	1	1
$X.13$	4	.	−4	.	4	.	.	.	.	.	.	−1	.	1	−1	−1	$-B$	B
$X.14$	4	.	−4	.	4	.	.	.	.	.	.	−1	.	1	−1	−1	B	$-B$
$X.15$	8	8	.	.	−1	.	.	.	.	.	.	8	−1	.	−1	−1	.	.
$X.16$	8	−8	.	.	−1	.	.	.	.	.	.	8	1	.	−1	−1	.	.
$X.17$	16	.	.	.	−2	.	.	.	.	.	.	−4	.	.	C	$/C$	.	.
$X.18$	16	.	.	.	−2	.	.	.	.	.	.	−4	.	.	$/C$	C	.	.

Note: Where $A = i$, $B = i\sqrt{5}$, and $C = \frac{1}{2}(1 + 3\sqrt{5})$.

Table B.6. Character table of $R = N_G(7A)$.

2	3	3	2	2	3	3	2	3	3	2	2	2	2	2	1	1	.	2	2	2	1	1	2	2
3	1	1	1	1	1	1	1	1	1	1	1	1	1	.	.	.	.	1	1	.	.	.	.	.
7	3	1	1	1	.	.	1	.	.	.	.	.	.	3	2	2	2	.	.	1	1	1	1	1
	1_a	2_a	2_b	2_c	3_a	3_b	4_a	6_a	6_b	6_c	6_d	6_e	6_f	7_a	7_b	7_c	7_d	12_a	12_b	14_a	14_b	14_c	28_a	28_b
$2P$	1_a	1_a	1_a	1_a	3_b	3_a	2_a	3_b	3_a	3_b	3_a	3_b	3_a	7_a	7_b	7_c	7_d	6_b	6_a	7_a	7_c	7_b	14_a	14_a
$3P$	1_a	2_a	2_b	2_c	1_a	1_a	4_a	2_a	2_a	2_b	2_b	2_c	2_c	7_a	7_b	7_c	7_d	4_a	4_a	14_a	14_b	14_c	28_a	28_b
$7P$	1_a	2_a	2_b	2_c	3_a	3_b	4_a	6_a	6_b	6_c	6_d	6_e	6_f	1_a	1_a	1_a	1_a	12_a	12_b	2_a	2_b	2_c	4_a	4_a
$X.1$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$X.2$	1	1	−1	1	1	1	−1	1	1	−1	−1	1	1	1	1	1	1	−1	−1	1	−1	1	−1	−1
$X.3$	1	1	1	−1	1	1	−1	1	1	1	1	−1	−1	1	1	1	1	−1	−1	1	1	−1	−1	−1
$X.4$	1	1	−1	−1	1	1	1	1	1	−1	−1	−1	−1	1	1	1	1	1	1	1	−1	−1	1	1
$X.5$	1	1	1	1	A	$/A$	1	A	$/A$	A	$/A$	A	$/A$	1	1	1	1	A	$/A$	1	1	1	1	1
$X.6$	1	1	−1	1	A	$/A$	−1	A	$/A$	$-A$	$-/A$	A	$/A$	1	1	1	1	$-A$	$-/A$	1	−1	1	−1	−1
$X.7$	1	1	1	−1	A	$/A$	−1	A	$/A$	A	$/A$	$-A$	$-/A$	1	1	1	1	$-A$	$-/A$	1	1	−1	−1	−1
$X.8$	1	1	−1	−1	A	$/A$	1	A	$/A$	$-A$	$-/A$	$-A$	$-/A$	1	1	1	1	A	$/A$	1	−1	−1	1	1
$X.9$	1	1	1	1	$/A$	A	1	$/A$	A	$/A$	A	$/A$	A	1	1	1	1	$/A$	A	1	1	1	1	1
$X.10$	1	1	−1	1	$/A$	A	−1	$/A$	A	$-/A$	$-A$	$/A$	A	1	1	1	1	$-/A$	$-A$	1	−1	1	−1	−1
$X.11$	1	1	1	−1	$/A$	A	−1	$/A$	A	$/A$	A	$-/A$	$-A$	1	1	1	1	$-/A$	$-A$	1	1	−1	−1	−1
$X.12$	1	1	−1	−1	$/A$	A	1	$/A$	A	$-/A$	$-A$	$-/A$	$-A$	1	1	1	1	$/A$	A	1	−1	−1	1	1
$X.13$	2	−2	.	.	$2/A$	$2A$	.	$-2/A$	$-2A$	.	.	.	.	2	2	2	2	.	.	−2	.	.	.	.
$X.14$	2	−2	.	.	$2A$	$2/A$	.	$-2A$	$-2/A$	.	.	.	.	2	2	2	2	.	.	−2	.	.	.	.
$X.15$	2	−2	.	.	2	2	.	−2	−2	.	.	.	.	2	2	2	2	.	.	−2	.	.	.	.
$X.16$	12	.	6	.	.	.	.	.	.	.	.	.	.	12	−2	5	−2	.	.	.	−1	.	.	.
$X.17$	12	.	−6	.	.	.	.	.	.	.	.	.	.	12	−2	5	−2	.	.	.	1	.	.	.
$X.18$	12	.	.	6	.	.	.	.	.	.	.	.	.	12	5	−2	−2	.	.	.	.	−1	.	.
$X.19$	12	.	.	−6	.	.	.	.	.	.	.	.	.	12	5	−2	−2	.	.	.	.	1	.	.
$X.20$	24	.	.	.	.	.	.	.	.	.	.	.	.	24	−4	−4	3	.	.	.	.	.	.	.
$X.21$	42	6	.	.	.	.	.	.	.	.	.	.	.	−7	.	.	.	.	.	−1	.	.	$-B$	B
$X.22$	42	6	.	.	.	.	.	.	.	.	.	.	.	−7	.	.	.	.	.	−1	.	.	B	$-B$
$X.23$	42	−6	.	.	.	.	6	.	.	.	.	.	.	−7	.	.	.	.	.	1	.	.	−1	−1
$X.24$	42	−6	.	.	.	.	−6	.	.	.	.	.	.	−7	.	.	.	.	.	1	.	.	1	1

Note: Where $A = \frac{1}{2}(-1 + i\sqrt{3})$ and $B = \sqrt{7}$.

References

[1] W. Bosma and J. Cannon, *MAGMA V2.11 Handbook*, University of Sydney, 2004. http://magma.maths.usyd.edu.au/magma/.

[2] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of Finite Groups* (Clarendon, Oxford, 1985).

[3] W. Feit, *The Representation Theory of Finite Groups*, North-Holland Mathematical Library, **25** (North-Holland Publishing Co., Amsterdam-New York, 1982).

[4] D. M. Goldschmidt, Automorphisms of trivalent graphs, *Ann. Math.* **111** (1980) 377–406.

[5] D. Gorenstein, *Finite Groups* (Harper and Row, New York, 1968).

[6] D. F. Holt, B. Eick and E. A. O'Brien, *Handbook of Computational Group Theory* (Chapman and Hall/CRC, Boca Raton, 2005).

[7] I. M. Isaacs, *Character Theory of Finite Groups* (Academic Press, New York, 1976).

[8] Z. Janko, A new finite simple group with abelian Sylow 2-subgroups and its characterization, *J. Algebra* **3** (1966) 147–186.

[9] M. Kratzer, Constructing pairs of compatible characters, *Algebra Colloq.* **10** (2003) 285–302.

[10] A. K. Lenstra, H. W. Lenstra and L. Lovász, Factoring polynomials with rational coefficients, *Math. Ann.* **261**(4) (1982) 515–534.

[11] G. O. Michler, Theory of finite simple groups, in preparation.

[12] G. O. Michler, The character values of multiplicity-free irreducible constituents of a transitive permutation representation, *Kyushu J. Math.* **55** (2001) 75–106.

[13] M. E. O'Nan, Some evidence for the existence of a new simple group, *Proc. London Math. Soc. (3)* **32** (1976) 421–479.

[14] M. Schönert *et al.*, *GAP 4.4 Manual (Groups, Algorithms and Programming)*, RWTH Aachen, Aachen, Germany, 2004. http://www.gap-system.org/.

[15] L. Soicher, A new existence and uniqueness proof for O'Nan's simple sporadic group, *Bull. London Math. Soc.* **22** (1990) 148–152.

[16] M. Suzuki, Characterizations of linear groups, *Bull. Amer. Math. Soc.* **75** (1969) 1043–1091.

[17] M. Suzuki, *Group Theory II* (Springer Verlag, New York, 1986).

[18] R. Wilson, P. Walsh, J. Tripp, I. Suleiman, S. Rogers, R. Parker, S. Norton, S. Nickerson, S. Linton, J. Bray and R. Abbott, *ATLAS of Finite Groups Representations*, http://web.mat.bham.ac.uk/atlas/v2.0/.