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Thompson's sporadic group uniquely determined by the centralizer of a 2-central involution

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Abstract

In this article we give a self contained existence and uniqueness proof for that sporadic simple group which was discovered by J.G. Thompson [J.G. Thompson, A simple subgroup of $E_8(3)$, in: N. Iwahori (Ed.), Finite Groups, Japan Soc. Promotion Science, Tokyo, 1976, pp. 113–116]. The centralizer H of a 2-central involution of that group has been described in terms of generators and relations by Havas, Soicher and Wilson in [G. Havas, L.H. Soicher, R.A. Wilson, A presentation for the Thompson sporadic simple group, in: W.M. Kantor, A. Seress (Eds.), Groups and Computation III, de Gruyter, Berlin, 2001, pp. 192–200]. Taking this presentation as the input of the second author's algorithm [G.O. Michler, On the construction of the finite simple groups with a given centralizer of a 2-central involution, J. Algebra 234 (2000) 668–693] we construct a simple subgroup $\mathfrak{G}$ of $\mathrm{GL}_{248}(11)$ which has a 2-central involution $\mathfrak{z}$ whose centralizer is isomorphic to H. In order to prove that the order of $\mathfrak{G}$ is $2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$, a faithful 143,127,000-dimensional permutation representation of this matrix group has been constructed on a supercomputer. In the second part of this article it is shown that any simple group G having a 2-central involution z with centralizer $C_G(z) \cong H$ is isomorphic to $\mathfrak{G}$. We construct its concrete character table as well.

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1. Introduction

A sporadic simple group of Th-type was originally discovered by J.G. Thompson. In [19] he announced the following theorem.

Theorem. *There is precisely one group E with the following properties*:

(a) *All involutions of E are conjugate.*
(b) *If z is an involution of E, $H = C_E(z)$ and $P = O_2(H)$, then P is extra special of order 2^9 and $H/P \cong A_9$.*

Among several other properties, he mentioned that

(c) $|E| = 2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$, *and that*
(d) *E has just one irreducible character of degree 248.*

Details of the proof for Thompson's theorem have not been published so far. In particular, the uniqueness question of the Thompson group has been considered to be an open problem.

It is the purpose of this article to show that the sporadic simple Thompson group G is uniquely determined up to isomorphism by an explicitly given presentation of the centralizer $C_G(z)$ of a 2-central involution z of G. This presentation is due to Havas, Soicher and Wilson [7]. It belongs to that sporadic simple group E which was originally constructed by J.G. Thompson and his collaborators, see [23]. A finite simple group G is called to be of Th-type if it has a 2-central involution z such that its centralizer $C_G(z) \cong H$.

In his preprint "A computer aided existence proof of Thompson's sporadic simple group" the first author took the presentation of H as the input of the second author's algorithm [12] and constructed a simple subgroup $\mathfrak{G}$ of $\mathrm{GL}_{248}(11)$ which he proved to have a 2-central involution $\mathfrak{z}$ with centralizer $C_{\mathfrak{G}}(\mathfrak{z}) \cong H$. Thus he gave a new existence proof for a simple group of Th-type. In the introduction of his preprint he wrote: "Recently, Michler [13] published a uniqueness criterion which allowed uniform uniqueness proofs for 11 other sporadic simple groups. It requires that for a given centralizer H of a 2-central involution z at least one simple group G with $C_G(z) \cong H$ has been constructed. Therefore a new construction of a finite simple group G with a 2-central involution z and $C_G(z) \cong 2^{1+8}.A_9$ is given in this article using the uniform construction method [12]." He continued: "It is the purpose of this and a subsequent paper to give a self contained proof for the existence and the uniqueness of Thompson's group." The referee of that article suggested that the first author combines both efforts in a revised version.

The first author's anticipated "uniqueness proof" quoted some essential results from Parrott's article [17] which assumes without explicit mention that up to isomorphism there is only one extension H of an extra-special group 2^{1+8} of $+$-type by an alternating group A_9. This is not the case as has been noted by the third author who asked then Derek Holt to provide explicit presentations for two nonisomorphic extensions H by means of the recent Cannon–Holt algorithm [2] implemented in MAGMA. The uniqueness proof given here is due to the second and third author. It realizes Weller's original plan without quoting results from [17]. Since it depends on the first author's existence proof we have written a joint article.

In Section 2 the explicit definition of the group H with center of even order is given, and all its important group and representation theoretic properties are derived. In particular, it is shown that a fixed Sylow 2-subgroup S of H contains a unique elementary abelian normal subgroup A of order 32 such that $D = N_H(A)$ is maximal among the normalizers in H of all noncyclic elementary abelian normal subgroups of S. Complete classifications of the conjugacy classes of H and D are given, and the character tables of both groups are calculated. They are stated in Sections 9 and 10, respectively.

Section 3 studies the fusion of the involutions of H in a finite simple group G of Th-type. Using Glauberman's Z^*-theorem it follows that any such group G has a unique conjugacy class of involutions. Applying the methods of [12] we also show that the normalizer $E = N_G(A)$ is uniquely determined up to isomorphism in any simple group G of Th-type. A presentation of E is given as well. A concrete character table of E has also been calculated. In fact, E is generated by D and the *Glauberman-element* fusing two involutions in the unique Klein four group V of S.

In Section 4 the first author's existence proof of a finite simple group $\mathfrak{G}$ of Th-type is described. He realizes $\mathfrak{G}$ inside $\mathrm{GL}_{248}(11)$. In order to calculate the order of $\mathfrak{G}$ the first author constructed a permutation representation of degree 143,127,000 on the supercomputer of Karlsruhe University and he verified the conditions of Gollan's double coset trick [6].

The second part of this article is devoted to the uniqueness proof. Starting from the structure of the nonisomorphic centralizers of representatives d_0, d_2 and d_4 of the three H-conjugacy classes of elements of order 3 we show in Section 5 that any simple group G of Th-type has 3 conjugacy classes of elements of order 3 as well, and that they are represented by these 3 elements of H. Furthermore, we derive for each normalizer $N_G(d_i)$ a uniquely determined presentation in terms of well defined generators and relations. The same has been done for the normalizer N_1 of the unique elementary abelian characteristic subgroup D_1 of order 9 in a fixed Sylow 3-subgroup of G. For each of these groups a complete classification of all its conjugacy classes is given, and a character table has been calculated. These tables are stated in Sections 9 and 10. We also determine the fusion of the conjugacy classes of these normalizers in any group G of Th-type. In particular, we have classified all 3-singular conjugacy classes of G.

In Section 6 we determine the 5- and 13-singular conjugacy classes of G. We also show that any simple group G of Th-type has either one or 2 conjugacy classes of elements of order 7. In each case, its Sylow 7-subgroups are shown to be of order 7^2. Presentations for the normalizers of the 7-elements in G are obtained as well. Unfortunately, at this stage we are not able to decide whether a Sylow 5-subgroup of G has order 5 or 125.

Using the second author's group order formula for abstract finite simple groups with a unique conjugacy class of involutions proved in [14] we are able to dismiss three of the 4 possibilities in Section 7. In particular, it follows that each finite simple group G of Th-type has a Sylow 5-subgroup of order 125, a single conjugacy class of elements of order 7, and that $|G| = 2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$.

Using our complete information about the conjugacy classes and the character tables of the normalizers of all elements of prime order of G and their fusion patterns in any simple group G of Th-type we are able to apply Brauer's characterization of characters in

Section 8. Thus we prove that such a simple group G has a unique irreducible character of degree 248. Since 11 does not divide $|G|$ it reduces to an irreducible 11-modular character of G which is realizable over GF(11). Now we can apply the uniqueness criterion of [13] and obtain the main theorem of this article which asserts that any simple group G of Th-type is isomorphic to the simple matrix group $\mathfrak{G} \leqslant \mathrm{GL}_{248}(11)$, see Theorem 8.2. As an application we also determine a complete system of representatives of all conjugacy classes of $\mathfrak{G}$ and its character table, see Section 11.

Concerning our notation and terminology we refer to the books by Feit [5], Isaacs [8] and Kurzweil and Stellmacher [11]. The algebra software packages MAGMA [1] and GAP [18] are frequently used.

2. The centralizer of a 2-central involution

In Lemma 2.2 of this section a presentation of the finite group H is given, which is assumed to be the centralizer of a 2-central involution in a simple group of Th-type. This presentation of H is due to Havas, Soicher and Wilson [7]. It is restated in Lemma 2.2 and provides the essential details of the following definition.

Definition 2.1. A finite simple group X is called to be of Th-*type* if it possesses a 2-central involution z such that $C_X(z) \cong H$, where H is the group of even order defined by generators and relations in Lemma 2.2.

Lemma 2.2. *Let $H := \langle z, c, d, e, s, t, u\rangle$ with the following set $\mathcal{R}(H)$ of defining relations*:

$$\begin{gathered}
z^2 = c^2 = d^2 = e^2 = (ze)^2 = (ce)^2 = (sd)^2 = 1,\\
z = (cd)^4, \qquad (de)^3 = u^4 = s^7 = t^3 = 1,\\
[s,z] = [s,c] = [t,z] = [t,c] = [t,d] = [t,e] = 1,\\
[e,s] = e^{s^3}, \qquad st = s^2, \qquad u^2 = zc, \qquad t^u = t^{-1},\\
[u,z] = [u,c] = [u,e] = 1,\\
\bigl(ded^u\bigr)^2 = 1, \qquad zc = (us)^3 = [u,s]^4,\\
\bigl(du^{s^2}\bigr)^4 = zcc^d c^{des^{-1}} c^{des^2}.
\end{gathered}$$

In H define the following elements:

$$\begin{gathered}
u_1 = z, \qquad u_2 = u^2, \qquad u_3 = c^{(u^d)}, \qquad u_4 = c^{(u^d s)}, \qquad u_5 = c^{(du^{-1}dst^{-1})},\\
v_1 = \bigl(uc^{(des^{-1})}uc^d\bigr)^s, \qquad v_2 = u\bigl(c^{de}c^d\bigr)^{(s^{-1})}u, \qquad v_3 = \bigl(s^{-1}\bigr)^t\bigl(s^2c^d\bigr)^e c^d e^{(s^{-1})},\\
v_4 = zc^d, \qquad x = t^{-1}u^s esc^{du}\bigl(u^{-1}\bigr)su^{-1}, \qquad y = ts^{-1}t\bigl(us^2u^d\bigr)^{se}sut,\\
w_1 = v_4u_3u_2xu_2v_3v_1u_5u_4v_4xu_2v_3v_2v_1v_4y^2xy^2u_3xy^2x^{-1}y^2,
\end{gathered}$$

$$\begin{aligned}
w_2 = {}& v_4u_3u_2xu_2v_3v_1v_3yv_4u_3u_2xu_2v_3v_1v_3yv_4u_3u_2xu_2v_3v_1v_3yv_4u_3v_4u_3u_2xu_2v_3\\
&\cdot v_1xu_2yxu_2v_3v_1xu_2v_3v_2v_1u_2^{-1}xu_4^{-1}y^{-1}v_3^{-1}v_1^{-1}v_3^{-1}u_2^{-1}x^{-1}u_2^{-1}u_3^{-1}yx^{-1}y^{-1}.
\end{aligned}$$

Then the following assertions hold:

(a) *The center of H is generated by $z = u_1$.*
(b) $S = \langle u_1, \dots, u_5, v_1, \dots, v_4, x^3, w_1, w_2\rangle$ *is a Sylow* 2*-subgroup of H with center* $Z(S) = \langle z\rangle$.
(c) $P = O_2(H) = \langle u_1, \dots, u_5, v_1, \dots, v_4\rangle = \{1\} \cup \{z\} \cup u_2^H$.
(d) $V = \langle z, u_2\rangle$ *is the uniquely determined normal Klein four subgroup of S.*
(e) $V = Z(C_H(V))$.
(f) *The Sylow* 2*-subgroup S of H has a unique elementary abelian normal subgroup* $A = \langle u_1, \dots, u_5\rangle$ *of order* 32 *such that* $D = N_H(A) \cong A.2^4.A_8$.
(g) *The Sylow* 2*-subgroup S of H has a unique elementary abelian normal subgroup B of order* 32 *such that* $N_H(B) \cong 2^{12}.\mathrm{PSL}_2(7)$.
(h) $D = \langle u_1, \dots, u_5, v_1, \dots, v_4, x, y\rangle$ *has the following set $\mathcal{R}(D)$ of defining relations*:

$$u_1^2 = \cdots = u_5^2 = [u_i, u_j] = 1 \quad \textit{for } 1 \leqslant i < j \leqslant 5, \qquad x^{24} = 1 = y^3,$$
$$v_1^2 = u_1, \qquad v_2^2 = u_1, \qquad v_3^2 = v_4^2 = u_1,$$
$$[u_1, v_1] = [u_1, v_2] = [u_1, v_3] = [u_1, v_4] = [v_1, u_2] = [v_1, u_3] = [v_1, u_4] = 1,$$
$$[v_1, u_5] = [v_2, u_3] = [v_2, u_5] = [v_3, u_3] = [v_3, u_4] = [v_3, u_5] = u_1,$$
$$[v_2, u_2] = [v_2, u_4] = [v_3, u_2] = 1,$$
$$[v_4, u_2] = [v_4, u_3] = [v_4, u_4] = [v_4, u_5] = u_1, \qquad [v_i, v_j] = 1 \quad \textit{for } 1 \leqslant i < j \leqslant 4,$$
$$x^6 u_2 u_4 v_1 v_4 = 1, \qquad (xy)^7 = 1,$$
$$[y, x]^2 u_1 u_2 u_3 u_4 u_5 v_1 = 1, \qquad \big(y x^{-2} y x^2\big)^2 u_2 u_3 = 1,$$
$$\big[y, x^3\big]^2 u_3 v_1 v_3 v_4 = 1, \qquad [u_1, x] = 1,$$
$$u_2^x u_1 u_2 u_5 = 1, \qquad u_3^x u_1 u_2 = 1, \qquad u_4^x u_1 u_3 = 1, \qquad u_5^x u_1 u_3 u_4 = 1,$$
$$[u_1, y] = 1, \qquad u_2^y u_1 u_5 = 1, \qquad u_3^y u_2 u_3 u_4 = 1,$$
$$u_4^y u_1 u_3 u_5 = 1, \qquad u_5^y u_2 u_5 = 1,$$
$$v_1^x u_4 u_5 v_2 v_3 = 1, \qquad v_2^x u_5 v_1 v_2 v_4 = 1,$$
$$v_3^x u_1 u_3 u_4 v_2 v_3 v_4 = 1, \qquad v_4^x u_1 u_3 v_1 v_2 v_3 v_4 = 1,$$
$$v_1^y u_1 u_5 v_2 v_4 = 1, \qquad v_2^y u_5 v_3 v_4 = 1,$$
$$v_3^y u_1 u_5 v_1 v_4 = 1, \qquad v_4^y u_1 u_2 u_5 v_1 v_2 v_3 = 1.$$

(i) *H has a faithful permutation representation of degree* 34,560 *with stabilizer* $H_1 = \langle e, s, t, u\rangle$.

Proof. Using the faithful permutation representation of H defined in (i) and MAGMA all assertions have been derived computationally from the given presentation of H. □

Lemma 2.3. *With the hypothesis of Lemma* 2.2 *the following statements hold*:

(a) $D = N_H(A) = \langle u_1, \dots, u_5, v_1, \dots, v_4, x, y\rangle = \langle x, y\rangle$.
(b) $H = \langle d, x, y\rangle$.
(c) *Representatives h_i of the* 52 *conjugacy classes h_i^H of H and the corresponding centralizer orders $|C_H(h_i)|$ are given in Table* 9.1.
(d) *The character table of H is given in Table* 10.1.
(e) *Representatives d_i of the* 48 *conjugacy classes d_i^D of D and the corresponding centralizer orders $|C_D(d_i)|$ are given in Table* 9.3.
(f) *The character table of D is given in Table* 10.3.

Proof. (a) Using the faithful permutation representation of H defined in Lemma 2.2(i) and MAGMA it has been checked that $D = N_H(A)$ is generated by x, y defined in Lemma 2.2.

(b) Similarly it has been verified that $H = \langle d, x, y\rangle$.

(c) The representatives h_i of the conjugacy classes h_i^H of H have been determined by means of algorithm [10].

(d) The character table of H has been calculated by means of the faithful permutation representation of H given in Lemma 2.2(i), MAGMA and algorithm of the first and second author [15].

(e) and (f) have been obtained, similarly. □

3. Conjugacy classes of elements of even order

In this section we study the fusion of the involutions of H in any finite simple group G of Th-type. Thus we obtain a classification of all conjugacy classes of G of elements of even order. Furthermore, it is shown that the normalizer $N_G(A)$ of the unique maximal elementary abelian normal subgroup A of the fixed Sylow 2-subgroup of H is uniquely determined up to isomorphism in all such simple groups G.

Lemma 3.1. *Let G be a finite simple group of* Th*-type with a* 2*-central involution z such that $C_G(z) = H$. Let A be the unique elementary abelian normal subgroup of order* 2^5 *in the Sylow* 2*-subgroup S of H defined in Lemma* 2.2. *Let $E = N_G(A)$ and $D = N_H(A)$. Then the following assertions hold*:

(a) *G has a unique conjugacy class z^G of involutions.*
(b) *There is an involution a in $E - D$ such that*

$$E = N_G(A) = \langle D, a\rangle = \langle u_1, \dots, u_5, v_1, \dots, v_4, x, y, a\rangle$$

has a set $\mathcal{R}(E)$ of defining relations consisting of $\mathcal{R}(D)$ and the following relations:

$$a^2 = 1, \qquad u_1^a = u_3u_4, \qquad u_2^a = u_1u_2u_3u_4,$$
$$[u_3, a] = 1, \qquad u_4^a = u_1u_3, \qquad [u_5, a] = 1,$$

Table 3.1

G	2_A			4_A		4_B				6_A	6_B			6_C	8_A	
H	2_a	2_b	2_c	4_a	4_c	4_b	4_d	4_e	4_f	6_a	6_b	6_d	6_e	6_c	8_a	8_c
E	2_a	2_b		4_a		4_b	4_c			6_b	6_a	6_c			8_a	

G	8_B			12_A		12_B		12_C	12_D	14_A			
H	8_a	8_d	8_c	12_a	12_d	12_b	12_e	12_c	12_f	14_a	14_b	14_c	
E	8_b	8_c			12_b	12_a			12_c	14_a	14_b	14_c	14_d

G	18_A	18_B	20_A	24_A	24_B	24_C	24_D	28_A		30_A	30_B	36_A	36_B
H	18_a	18_b	20_a	24_a	24_b	24_c	24_d	28_a		30_a	30_b	36_a	36_b
E				24_b	24_a			28_a	28_b	30_b	30_a		

$$\begin{aligned}
ax^3a &= v_2v_1v_4ax^3,\\
\left(x^6\right)v_1v_3y &= xy^2x^{-1}y^2x^{-1}yx^2y^2x\left(y^2x^{-1}\right)^2yx^{-2}y^2xy\left(yx^{-1}\right)^3xyx^{-1}yx^{-2},\\
x^ax^{-2}yay &= xy^2x^{-1}y^2x^{-1}yx^2y^2xy^2x^{-1}y^2x^{-1}yx^{-2}y^2xy\left(xy^2\right)^2\\
&\quad \cdot xyx^{-1}yx^2y^2x\left(y^2x^{-1}\right)^2yx^{-4}y^2xy^2x^{-1}\left(xy^2\right)^2y^2xyx^2u_1.
\end{aligned}$$

(c) *E has a faithful permutation representation of degree* 7440 *with stabilizer*

$$\begin{aligned}
M_1 = \big\langle & xv_3ax(yxy)^{-1}, v_1a(u_1v_1)^{-1}, v_2yx(v_1yv_3)^{-1}, u_3^x, u_1u_2, u_3, u_5,\\
& u_1u_4, a, v_3a(u_1v_3)^{-1}\big\rangle.
\end{aligned}$$

(d) *The amalgam $H \leftarrow D \rightarrow E$ has Goldschmidt index* 1.

(e) *Representatives m_i of the* 41 *conjugacy classes m_i^E of H and the corresponding centralizer orders $|C_E(m_i)|$ are given in Table* 9.2.

(f) *The character table of E is given in Table* 10.2.

(g) *G has* 27 *z-special conjugacy classes of elements of even order represented by the H-classes*:

$$2_a, 4_a, 4_b, 6_a, 6_b, 6_c, 8_a, 8_b, 10_a, 12_a, 12_b, 12_c, 12_f, 14_a, 18_a, 18_b, 20_a,$$

$$24_a, 24_b, 24_c, 24_d, 28_a, 30_a, 30_b, 36_a, 36_b \textit{ and } 36_c.$$

As G-classes they are denoted by: $2_A, 4_A, 4_B, 6_A, 6_B, 6_C$, *etc.*

(h) *Using the classification of the conjugacy classes of elements of* 2*-power order of $H = C_G(z)$ and $E = N_G(A)$ given in Tables* 9.1 *and* 9.2, *respectively their fusion patterns into the conjugacy classes of G are given in Table* 3.1.

Proof. (a) By Lemma 2.2(d) $V = \langle z, u_2\rangle$ is the unique normal Klein four subgroup of the Sylow 2-subgroup S defined in Lemma 2.2(b). By Table 9.1 H has 3 conjugacy classes of involutions represented by z, u_2 and d with centralizer orders $|C_H(z)| = |H|$, $|C_H(u_2)| = 2^{15} \cdot 3^4 \cdot 5 \cdot 7$ and $|C_H(d)| = 2^{10} \cdot 3$.

Suppose that $u_2 \notin z^G$, then $d \in z^G$ by Glauberman's Z^*-theorem. Let $P = O_2(H)$ and $Q = C_P(d)$. Then using the faithful permutation representation of H given in Lemma 2.2(i) and MAGMA it follows that $|Q| = 2^4$. By Lemma 2.2(c) $P = O_2(H) = \{1\} \cup \{z\} \cup u_2^P$. Hence $Q = \langle u_2^H \cap Q \rangle$. Let $d = z^g$ for some $g \in G$. Then $H^g = C_G(j) = J$ and $(u_2^H \cap Q) \leqslant O_2(J)$. Hence $z \in u_2^H \cap Q \leqslant O_2(J)$. Another application of Lemma 2.2(c) yields that $O_2(J) = \{1\} \cup \{d\} \cup u_2^J$. As $z \neq d$ it follows that $z \in u_2^J$, a contradiction. Thus $u_2 \in z^G$, and there is a $g \in G - H$ such that $z^g = u_2$. Now Proposition 5.2.2 of [14] implies that $g \in N_G(V)$.

By Lemma 2.2(f) $A = \langle z, u_2, u_3, u_4, u_5 \rangle$ is the unique maximal elementary abelian normal subgroup of the Sylow 2-subgroup S of H. Another application of MAGMA yields that $A = O_2[C_H(V)]$ and $C_H(A) = A$. As $C_G(V) = C_H(V) \lhd N_G(V)$ and A is a characteristic subgroup of $C_H(V)$ it follows that $N_G(V) \leqslant N_G(A)$. Hence $g \in N_G(V) - N_H(V)$ belongs to $N_G(A) - N_H(A)$. Since $N_H(A)/C_G(A) \cong 2^4.A_8$ by Lemma 2.2 and $2^4.A_8 \cong 2^4\mathrm{GL}_4(2)$ is a maximal subgroup of $\mathrm{GL}_5(2)$ it follows that $N_G(A)/A \cong \mathrm{GL}_5(2)$.

Using MAGMA and the faithful permutation representation of H given in Lemma 2.2(i) it has been checked that $D = N_H(A)$ is a nonsplit extension of $D/A \cong 2^4.A_8$ by $A \cong 2^5$. Hence the theorem of Gaschütz implies that $E = N_G(A)$ is a nonsplit extension of A by $E/A \cong \mathrm{GL}_5(2)$.

Let $\eta : D \to \mathrm{GL}_5(2)$ be the representation of $D = \langle x, y, v_1, v_2, v_3, v_4, A \rangle$ afforded by the conjugate action of D on A with respect to the basis $\mathbb{A} = \{u_1, \ldots, u_5\}$. Then Lemma 2.2(h) implies that

$$\eta(x) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & . & . \\ . & . & . & 1 & 1 \\ . & . & . & . & 1 \\ . & 1 & . & . & . \end{pmatrix}, \qquad \eta(y) = \begin{pmatrix} 1 & 1 & . & 1 & . \\ . & . & 1 & . & 1 \\ . & . & 1 & 1 & . \\ . & . & 1 & . & . \\ . & 1 & . & 1 & 1 \end{pmatrix},$$

$$\eta(v_1) = \begin{pmatrix} 1 & . & . & . & 1 \\ . & 1 & . & . & . \\ . & . & 1 & . & . \\ . & . & . & 1 & . \\ . & . & . & . & 1 \end{pmatrix}, \qquad \eta(v_2) = \begin{pmatrix} 1 & . & 1 & . & 1 \\ . & 1 & . & . & . \\ . & . & 1 & . & . \\ . & . & . & 1 & . \\ . & . & . & . & 1 \end{pmatrix},$$

$$\eta(v_3) = \begin{pmatrix} 1 & . & 1 & 1 & 1 \\ . & 1 & . & . & . \\ . & . & 1 & . & . \\ . & . & . & 1 & . \\ . & . & . & . & 1 \end{pmatrix} \quad \text{and} \quad \eta(v_4) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ . & 1 & . & . & . \\ . & . & 1 & . & . \\ . & . & . & 1 & . \\ . & . & . & . & 1 \end{pmatrix}.$$

Let $\hat{\eta}$ be the extension of η to $E = N_G(A) = \langle x, y, g \rangle$. Since $\hat{\eta}(E) = \mathrm{GL}_5(2)$ the image $\hat{\eta}(g)$ of the new generator g of E can be exchanged by any other matrix $\eta(a)$. Using MAGMA it follows that $\eta(E) = \mathrm{GL}_5(2) = \langle \eta(x), \eta(y), \eta(a) \rangle$ for the matrix

$$\eta(a)=\begin{pmatrix} . & 1 & . & 1 & . \\ . & 1 & . & . & . \\ 1 & 1 & 1 & 1 & . \\ 1 & 1 & 1 & . & . \\ . & . & . & . & 1 \end{pmatrix} \quad \text{of order 2.}$$

Furthermore, $\bar{E}=\hat{\eta}(E)=\mathrm{GL}_5(2)=\langle\eta(x),\eta(y),\eta(a)\rangle$ has the following set $\mathcal{R}(\bar{E})$ of defining relations:

$$\begin{aligned}
\eta(a)^2 &= 1,\\
\eta(a)\eta(x)^3\eta(a) &= \eta(v_2)\eta(v_1)\eta(v_4)\eta(a)\eta(x)^3,\\
\bigl(\eta(x)^6\bigr)^{\eta(a)}\eta(v_1)\eta(v_3)\eta(y) &= \eta(x)\eta(y)^2\eta(x)^{-1}\eta(y)^2\eta(x)^{-1}\eta(y)\eta(x)^2\eta(y)^2\eta(x)\\
&\quad\cdot\bigl(\eta(y)^2\eta(x)^{-1}\bigr)^2\eta(y)\eta(x)^{-2}\eta(y)^2\eta(x)\eta(y)\\
&\quad\cdot\bigl(\eta(y)\eta(x)^{-1}\bigr)^3\eta(x)\eta(y)\eta(x)^{-1}\eta(y)\eta(x)^{-2},\\
\eta(x)^{\eta(a)}\eta(x)^{-2}\eta(y)\eta(a)\eta(y) &= \eta(x)\eta(y)^2\eta(x)^{-1}\eta(y)^2\eta(x)^{-1}\eta(y)\eta(x)^2\eta(y)^2\eta(x)\eta(y)^2\\
&\quad\cdot\eta(x)^{-1}\eta(y)^2\eta(x)^{-1}\eta(y)\eta(x)^{-2}\eta(y)^2\eta(x)\eta(y)\\
&\quad\cdot\bigl(\eta(x)\eta(y)^2\bigr)^2\cdot\eta(x)\eta(y)\eta(x)^{-1}\eta(y)\eta(x)^2\eta(y)^2\eta(x)\\
&\quad\cdot\bigl(\eta(y)^2\eta(x)^{-1}\bigr)^2\eta(y)\eta(x)^{-4}\eta(y)^2\eta(x)y^2\eta(x)^{-1}\\
&\quad\cdot\bigl(x\eta(y)^2\bigr)^2\eta(y)^2\eta(x)\eta(y)\eta(x)^2.
\end{aligned}$$

As $\eta(a)$ is a well determined matrix in $\mathrm{GL}_5(2)$ its operation of the GF(2)-vector space A is given explicitly. Let a be an inverse image of $\eta(a)$ in G. Then a operates on A as $\eta(a)$. Hence the following relations also hold:

$$\begin{aligned}
&a^2=1, && u_1^a=u_3u_4, && u_2^a=u_1u_2u_3u_4,\\
&[u_3,a]=1, && u_4^a=u_1u_3, && [u_5,a]=1.
\end{aligned}$$

Multiplying the last equation by u_1 and replacing all $\eta(v_i)$, $\eta(x)$, $\eta(y)$ by v_i, x and y, respectively, one obtains the set $\mathcal{R}(E)$ of relations given in assertion (b). Using the Todd–Coxeter algorithm implemented in MAGMA it has been checked that $\mathcal{R}(E)$ is a defining set of relations for a nonsplit extension

$$1\to A\to E\to \mathrm{GL}_5(2)\to 1,$$

and that the subgroup

$$\begin{aligned}
M_1 = \bigl\langle &xv_3ax(yxy)^{-1}, v_1a(u_1v_1)^{-1}, v_2yx(v_1yv_3)^{-1}, u_3^x, u_1u_2, u_3, u_5,\\
&u_1u_4, a, v_3a(u_1v_3)^{-1}\bigr\rangle
\end{aligned}$$

of E defines a faithful permutation representation of E with degree 7440.

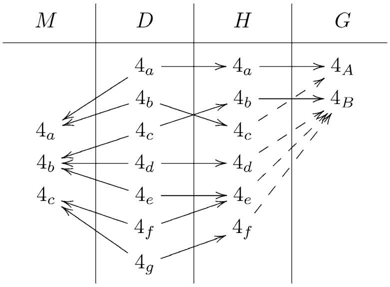

Fig. 3.1.

Using also the faithful permutation representation of H defined in Lemma 2.2(i) and Algorithm 7.1.10 of [14] it has been checked that the Goldschmidt index of the amalgam $H \leftarrow D \rightarrow E$ is one. Hence $E = N_G(A)$ is uniquely determined up to isomorphism for any finite simple group G of Th-type by Theorem 7.4.2 of [14].

Using the faithful permutation representation of M of degree 7440 and MAGMA the representatives m_i of the 41 conjugacy classes m_i^E of E have been calculated by means of Kratzer's algorithm [10]. The results are stated in Table 9.2.

It follows that $E = \langle x, y, a\rangle$ has 2 conjugacy classes of involutions represented by $z = x^{12}$ and a. Since d is not conjugate to z in H, but H-conjugate to $(xyxy^2)^2$ by Tables 9.1 and 9.2, and as $(xyxy^2)^2 \in D$ and a are conjugate in E, it has been proved that G has a unique conjugacy class of involutions.

The fusion patterns of D-classes into the ones of H and M are computed with MAGMA [1]. For example, the fusion graph for the elements of order 4 looks as shown in Fig. 3.1.

4. Existence proof of Th inside $\mathrm{GL}_{248}(11)$

In this section we describe the first author's construction of a simple group of Th-type by means of the second author's algorithm [12]. For that application the following notation and definitions are restated.

In the following text the set of all faithful characters of the finite group U is denoted by $f\,\mathrm{char}_{\mathbb{C}}(U)$, and $mf\,\mathrm{char}_{\mathbb{C}}(U)$ denotes the set of all multiplicity-free faithful characters of U.

Definition 4.1. Let U_1, U_2 be a pair of finite groups intersecting in D. Then

$$\Sigma = \big\{(\nu, \omega) \in mf\,\mathrm{char}_{\mathbb{C}}(U_1) \times mf\,\mathrm{char}_{\mathbb{C}}(U_2) \mid \nu|_D = \omega|_D\big\}$$

is called the set of *compatible pairs of multiplicity-free faithful characters* of U_1 and U_2.

For each $(\nu, \omega) \in \Sigma$ the integer $n = \nu(1) = \omega(1)$ is called the *degree* of the compatible pair (ν, ω).

Theorem 4.2. *Let A be the unique elementary abelian normal subgroup of order 2^5 in the Sylow 2-subgroup S of H defined in Lemma 2.2. Let $D = N_H(A) = \langle u_1, \ldots, u_5, v_1, \ldots, v_4, x, y\rangle$ with set $\mathcal{R}(D)$ of defining relations given in Lemma 2.2. Let $E = \langle D, a\rangle$ be the finite group with set $\mathcal{R}(E)$ of defining relations consisting of $\mathcal{R}(D)$ and the additional relations given in Lemma 3.1. Then the following statements hold*:

(a) *The smallest degree of a nontrivial pair $(\chi, \tau) \in mf\,\mathrm{char}(H) \times mf\,\mathrm{char}(E)$ of compatible characters is 248.*

(b) *There are two compatible pairs $(\chi, \tau) \in mf\,\mathrm{char}(H) \times mf\,\mathrm{char}(E)$ of degree 248 of the groups $H = \langle d, x, y\rangle$ and $E = \langle a, x, y\rangle$*:

$$\chi_{14} + \chi_{\mathbf{18}} = \tau_{\mathbf{6}},$$
$$\chi_{14} + \chi_{\mathbf{19}} = \tau_{\mathbf{6}}$$

with common restrictions

$$\tau_6|_D = \psi_{24} + \psi_{27}.$$

(c) *Let $\mathfrak{V}$ and $\mathfrak{W}$ be the up to isomorphism uniquely determined faithful semisimple multiplicity-free 248-dimensional modules of H and E over $F = \mathrm{GF}(11)$ corresponding to the first compatible pair $\chi = \chi_{14} + \chi_{\mathbf{18}}$ and $\tau = \tau_{\mathbf{6}}$, respectively.*

Let $\kappa_{\mathfrak{V}} : H \to \mathrm{GL}_{248}(11)$ and $\kappa_{\mathfrak{W}} : N \to \mathrm{GL}_{248}(11)$ be the representations of H and E afforded by the modules $\mathfrak{V}$ and $\mathfrak{W}$, respectively.

Let $\mathfrak{x} = \kappa_{\mathfrak{V}}(x)$, $\mathfrak{y} = \kappa_{\mathfrak{V}}(y)$, and $\mathfrak{d} = \kappa_{\mathfrak{V}}(d) \in \kappa_{\mathfrak{V}}(H) \leqslant \mathrm{GL}_{248}(11)$. Then the following assertions hold:

(1) *$\mathfrak{V}|_D \cong \mathfrak{W}|_D$, and there is a transformation matrix $\mathcal{T} \in \mathrm{GL}_{248}(11)$ such that*

$$\mathfrak{x} = \mathcal{T}^{-1}\kappa_{\mathfrak{W}}(x)\mathcal{T} \quad \text{and} \quad \mathfrak{y} = \mathcal{T}^{-1}\kappa_{\mathfrak{W}}(y)\mathcal{T}.$$

Let $\mathfrak{a} = \mathcal{T}^{-1}\kappa_{\mathfrak{W}}(a)\mathcal{T} \in \mathrm{GL}_{248}(11)$.

(2) *$\mathfrak{G} = \langle \mathfrak{x}, \mathfrak{y}, \mathfrak{d}, \mathfrak{a}\rangle$ has a subgroup $\mathfrak{T}$ stabilizing a 52-dimensional subspace $\mathfrak{U}$ of $\mathfrak{V} = F^{248}$ such that $\mathfrak{T}$ is the stabilizer of the permutation group action of $\mathfrak{G}$ on the $\mathfrak{G}$-orbit $\mathfrak{U}^{\mathfrak{G}}$ of degree 143,127,000, and*

$$|\mathfrak{T}| = 2^{12} \cdot 3^5 \cdot 7^2 \cdot 13.$$

(3) *$\mathfrak{G} = \langle \mathfrak{x}, \mathfrak{y}, \mathfrak{d}, \mathfrak{a}\rangle$ is a finite simple group with 2-central involution $\kappa_{\mathfrak{V}}(z)$ such that*

$$C_{\mathfrak{G}}\bigl(\kappa_{\mathfrak{V}}(z)\bigr) = \kappa_{\mathfrak{V}}(H), \quad \text{and} \quad |\mathfrak{G}| = 2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31.$$

(4) *Its generating matrices can be down loaded from the first author's web side.*

Proof. (a) The character tables of the groups H, D and $E = N_G(A)$ are known by Lemmas 2.3 and 3.1, respectively. They are stated in Section 10. Furthermore, Lemma 3.1 asserts that the amalgam $H \leftarrow D \rightarrow E$ has Goldschmidt index one. Therefore the free product $P = H *_D N$ with amalgamated subgroup D is uniquely determined up to isomorphism by Corollary 7.1.9 of [14]. The fusion of the conjugacy classes of D into the ones of the two groups H and E has been determined in Lemma 3.1. Therefore Kratzer's algorithm [10] can be applied to check that there are no compatible pairs $(\chi, \tau) \in mf\,\mathrm{char}(H) \times mf\,\mathrm{char}(E)$ of degree smaller than 248.

(b) The application of Kratzer's deterministic algorithm also yields that there are exactly two compatible pairs of faithful semisimple characters of degree 248. Their irreducible constituents are stated in the assertion.

(c) By Lemma 2.2 the group $H = \langle x, y, d\rangle$ has a faithful permutation representation of degree 34,560. Let P_H be the corresponding FH-module over $F = \mathrm{GF}(11)$. As $gcd(|H|, 11) = 1$ its endomorphism ring $\mathrm{End}_{FH}(P_H)$ is semisimple by Maschke's theorem. Therefore we can apply first and second authors algorithm from [15] to calculate the centrally primitive idempotents e_2, e_{18}, e_{19}, e_{37} of $\mathrm{End}_{FH}(P_H)$ corresponding to the irreducible characters χ_2, χ_{18}, χ_{19}, χ_{37} in $\mathrm{Irr}_{\mathbb{C}}(H)$ occurring as irreducible constituents in the permutation character of P_H, respectively. Let $\mathfrak{V}_i = e_i \cdot P_H$ for $i = 2, 18, 19, 37$, and let $\mathfrak{V}_{14}$ be the irreducible FH-module belonging to the irreducible character χ_{14} of H. Forming now the tensor product $\mathfrak{V}_2 \otimes \mathfrak{V}_{37}$ of degree 7680 and applying then Parker's Meat-Axe algorithm [16] implemented in MAGMA one obtains an irreducible constituent which is isomorphic to $\mathfrak{V}_{14}$.

The centralizer $C = C_E(x^8)$ of x^8 in the group $E = \langle x, y, a\rangle$ has order $|C| = 2160$. It has been checked by the first author that 1_C^E is a faithful permutation character of degree 79,360. Let P_E be the corresponding FE-module over $F = \mathrm{GF}(11)$. As $gcd(|E|, 11) = 1$ the endomorphism ring $\mathrm{End}_{FE}(P_E)$ is also semisimple. Another application of algorithm [15] yields the centrally primitive idempotent f_6 of $\mathrm{End}_{FE}(P_E)$ corresponding to the irreducible constituent τ_6 of the permutation character of P_E. Hence $\mathfrak{W} = f_6 \cdot P_E$ is the simple FE-module corresponding to τ_6 in $\mathrm{Irr}_{\mathbb{C}}(E)$. Thus all irreducible constituents of the two pairs of semisimple faithful multiplicity-free FH- and FE-modules corresponding to the two compatible pairs determined in (b) have been constructed.

Let $\mathfrak{V} = \mathfrak{V}_{14} \oplus \mathfrak{V}_{18}$. Then $\mathfrak{V}|_D \cong \mathfrak{W}|_D$ by (b). In each restricted FD-module $\mathfrak{V}|_D$ and $\mathfrak{W}|_D$ we fix an F-vector space basis $\mathbb{B}$ and $\mathbb{S}$, respectively. Let $\mathfrak{x}'$, $\mathfrak{y}'$ and $\mathfrak{a}'$ be the matrices of the generators x, y and a of E with respect to the basis $\mathbb{S}$. Let $\mathfrak{x}$, $\mathfrak{y}$ and $\mathfrak{d}$ be the matrices of the generators x, y and d of H with respect to the basis $\mathbb{B}$. Applying then Parker's Isomorphism Test described in [16] one obtains a transformation matrix $\mathcal{T} \in \mathrm{GL}_{248}(11)$ corresponding to the base change $\mathbb{S} \to \mathbb{B}$ such that

$$\mathfrak{x} = \mathcal{T}^{-1}\mathfrak{x}'\mathcal{T} \quad \text{and} \quad \mathfrak{y} = \mathcal{T}^{-1}\mathfrak{y}'\mathcal{T}.$$

From (b) and Thompson's theorem [20] follows that the free product $P = H *_D N$ with amalgamated subgroup D has a unique simple 248-dimensional module over $F = \mathrm{GF}(11)$. Let κ be the irreducible representation of P afforded by this module. Let $\kappa(P) = \mathfrak{G}$. Then Thompson's theorem also asserts that

$$\mathfrak{G} = \langle \mathfrak{x}, \mathfrak{y}, \mathfrak{a}, \mathfrak{d}\rangle \leqslant \mathrm{GL}_{248}(11),$$

and that this finite group has a 248-dimensional irreducible representation. Its generating matrices $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{a}$ and $\mathfrak{d}$ are too large to be printed in this article. They can be obtained from the first author by e-mail.

Of course, all the steps of this construction can also be done for the second compatible pair

$$\chi_{14} + \chi_{\mathbf{19}} = \tau_{\mathbf{6}}.$$

All we have to do is to replace the FH-module $\mathfrak{V} = \mathfrak{V}_{14} \oplus \mathfrak{V}_{18}$ by $\mathfrak{V} = \mathfrak{V}_{14} \oplus \mathfrak{V}_{19}$. However, the matrix group $\mathfrak{G}*$ we obtain this time does not satisfy the Sylow 2-subgroup test of step 5c) of algorithm [12]. Constructing random words in the generators of $\mathfrak{G}*$ the first author has found elements of order 16 in this group. But H does not have elements of order 16 by Table 9.1. In the remaining case he was not able to find such elements. Therefore, we have now to give a proof for the correct order of the Sylow 2-subgroups of $\mathfrak{G}$.

By technical reasons steps 6 and 7 of algorithm [12] cannot be applied directly to determine the order of the matrix group $\mathfrak{G}$. Therefore the first author has constructed a suitable subgroup as follows.

In order to simplify our notation in the remainder of this proof we replace the Gothic letters $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{a}$, $\mathfrak{d}$ and by x, y, a, d, respectively. In particular, $\mathfrak{G} = \langle x, y, a, d\rangle \leqslant \mathrm{GL}_{248}(11)$ is also denoted by G.

In $G = \langle d, x, y, a\rangle$ define the following elements:

$$\begin{aligned}
b_M &= yay(xy)^2a^{-1}xy^{-1}axax^2yxayx^2axa^{-1}x^{-3}a^{-1}(xa)^2x^3 \\
&\quad \cdot y^{-1}x^{-2}a^{-1}x^{-1}a^{-1}yx^{-1}a\bigl(y^{-1}x^{-1}\bigr)^2y^{-1}a^{-1}y^{-1}x, \\
b &= \bigl(b_M^2dab_M\bigr)^{13}, \\
b_T &= \bigl(\bigl((db)^2db^2(db)^4db^2(db)^2b\bigr)^5\bigr)^{(db)^{11}}, \\
c &= (db_T)^2(b_Tdb_T)^2(db_T)^3.
\end{aligned}$$

Using the Parker's algorithm [16] it follows that the subgroup $T = \langle d, b_T\rangle$ fixes a 52-dimensional subspace U of the FG-module $V = V_{14} \oplus V_{18}$. The multiplication of the generating matrices of G by the vectors v of V defines a permutation action of G on the G-orbit $\Omega = U^G$. Now one constructs a permutation representation $\pi : G \to S_n$ of degree $n = 143{,}127{,}000$ with stabilizer $\mathrm{Stab}_G(U) := S \geqslant T$. The points i of Ω correspond to the cosets Sg_i of S in G for $i = 1, 2, \ldots, 143{,}127{,}000$, where $g_1 = 1$. This permutation module has been established by the first author on a supercomputer at the University of Karlsruhe. In order to save memory, he only stored suborbits of the element c of order 13 in a hash table using the methods described in his article [22].

Clearly, the subgroup $T = \langle d, b_T\rangle$ of G acts on Ω. Tracing the Schreier vector of G on Ω at the starting points of the T-orbits of Ω one gets the double coset representatives of T in G. The first author computed also a base and strong generating set for $\pi(G)$. It follows that:

Table 4.1

Representative		$\lvert Tr_iT\rvert$	o	Point
r_1	1	1	1	23, 078, 732
r_2	$bc^3b_Tcb_Tc^4b_Tc^{10}b_Tc^{12}b_Tc^8b_Tc^5b_Tc^7b_T^2bc^5$	17,199	14	37, 140, 703
r_3	$bcb_T^2cb_Tc^6b_Tc^2b_Tc^2b_Tc^{10}b_Tc^4b_Tc^4b_Tc^9bc$	45,864	20	41, 102, 441
r_4	$bc^5b_Tc^8b_Tcb_Tc^{11}b_Tc^2b_Tc^4b_Tc^6b_Tc^{11}b_Tc^3bc^6$	179,712	20	90, 617, 375
r_5	$bc^{12}b_Tc^{11}b_Tc^2b_T^2cb_Tc^3b_Tc^3b_Tbc^9$	1,304,576	19	14, 019, 617
r_6	$bc^2b_Tc^{11}b_Tc^8b_Tc^7b_Tc^3b_Tc^4b_Tc^8bc^8$	2,201,472	31	116, 942, 791
r_7	$bc^{10}b_Tc^{12}b_Tcb_Tc^2b_Tc^{11}b_Tc^6b_Tc^5b_Tc^2b_Tc^8bc^8$	5,031,936	28	15, 874, 808
r_8	$bc^{10}b_Tc^{12}b_Tcb_Tc^2b_Tc^{11}b_Tc^6b_Tc^5b_Tc^2b_Tc^6bc^{10}$	8,128,512	24	30, 184, 585
r_9	$r_8b_Tc^7b_Tc^{10}b_Tc^4b_Tcb_Tc^2b_Tc^{10}bc^8$	8,805,888	15	142, 921, 133
r_{10}	$bc^{10}b_Tc^{12}b_Tcb_Tc^2b_Tc^{11}b_Tc^6b_Tc^5b_Tc^2b_Tc^{12}bc^2$	11,741,184	18	105, 310, 456
r_{11}	bc^5	105,670,656	12	48, 089, 759

Table 4.2

Coset representative		Point $\beta_i = V\gamma_i$	o	Fixed by	$\lvert (V\gamma_i)^{\langle s_1,\dots,s_4\rangle}\rvert$
γ_1	1	23, 078, 732	1	s_2, s_3, s_4	143, 127, 000
γ_2	r_{11}	48, 089, 759	12	s_3, s_4	105, 670, 656
γ_3	$r_{11}c$	48, 089, 760	21	–	6

Strong generator		o	Fixes
s_1	b	3	–
s_2	b_T	3	$V\gamma_1$
s_3	$c^3b_Tc^4b_Tcb_Tc^{10}(c^{11}b_Tc^7b_Tcb_Tc^{12})^{-1}$	3	$V\gamma_1, V\gamma_2$
s_4	$c^2b_Tc^7b_T^2c^3b_Tc^3(b_Tc^6b_Tc^{10}b_Tcb_Tc^7)^{-1}$	2	

(α) G has 11 double cosets Tr_iT with representatives r_i of order $o = o(r_i)$, see Table 4.1.
(β) The element c of T has exactly 3 fix points:

$$\begin{aligned}
23,078,732 &\mathrel{\hat{=}} V,\\
38,447,957 &\mathrel{\hat{=}} Vr_8c^5b_Tc^6b_Tcb_Tc^3b_Tcb_Tc^7b_T \quad \text{and}\\
75,513,376 &\mathrel{\hat{=}} Vr_8b_Tc^9b_Tcb_Tc^2b_Tc^{10}b_Tc^2b_Tc^{11}b_T.
\end{aligned}$$

(γ) $\pi(G)$ has a base and strong generators given in Table 4.2.
(δ) $T = \langle s_2, s_3, s_4\rangle = \langle b_T, c\rangle$ and $G = \langle T, b\rangle$.

Statement (δ) holds by the following equations which have been verified computationally:

$$\begin{aligned}
d = {} & s_3^2s_2s_3s_2s_3^2s_2s_3s_2^2s_3^2s_2s_4s_2s_3^2s_2^2s_3s_2s_3^2s_2s_3s_2s_3^2s_2^2s_4s_3^2s_2^2s_3^2s_4s_2^2s_4s_2^2s_3s_2\\
& \cdot s_4s_2s_4s_3s_2s_3s_4s_2s_3s_2^2s_3^2s_2^2s_3s_2^2s_3^2s_2s_3s_2^2s_4s_2^2s_3s_2s_3^2s_2^2s_3s_2^2s_3^2
\end{aligned}$$

$$\begin{aligned}
&\cdot s_2^2s_3^2s_4s_3^2s_2s_3s_2s_3^2s_2s_3s_2s_2s_3^2s_2s_4s_2s_3^2s_2^2s_3s_2s_3^2s_2s_3s_2s_3^2s_2^2s_4s_3^2s_2^2s_3^2s_2s_4s_3\\
&\cdot s_2s_3s_4s_2s_3s_2^2s_3^2s_2^2s_3s_2^2s_3^2s_2s_3s_2^2s_4s_2^2s_3s_2^2s_2^2s_3^2s_2^2s_3s_2^2s_3^2s_2^2s_3^2s_4s_2s_4\\
&\cdot s_3s_2s_3s_4s_2s_3s_2^2s_3^2s_2^2s_3s_2^2s_3^2s_2s_3s_2^2s_4s_2^2s_3s_2s_3^2s_2^2s_3s_2^2s_3^2s_2^2s_3^2s_2^2,\\
c ={}& s_3s_4s_2^2s_3^2s_4s_3s_2s_4s_2s_4s_3^2s_4s_2s_3^2s_4s_2s_3s_2s_3^2s_2s_3^2s_4s_2^2s_4s_2s_4s_2s_3\\
&\cdot s_2^2s_4s_2s_3s_2s_3s_2^2s_4s_3s_4s_2^2s_4s_2^2s_3^2s_4s_3s_4s_2s_4s_3s_2^2s_3s_2^2s_3^2s_2^2s_4s_3s_2^2s_4s_3s_4s_2^2s_4\\
&\cdot s_2^2s_3^2s_4s_3s_2s_3^2s_4s_3s_2s_4s_2s_4s_3^2s_4s_2s_3^2s_2^2s_3^2s_2^2s_4s_2s_3^2s_2^2s_3^2\\
&\cdot s_2^2s_3^2s_4s_3s_2s_4s_2s_4s_3^2s_4s_2s_3^2s_4s_2s_3s_2s_3^2s_2s_3^2s_4s_2^2s_4s_2s_3s_2^2s_3^2s_4\\
&\cdot s_3s_2s_4s_2s_4s_3^2s_4s_2s_3^2s_2^2s_3^2s_2^2s_4s_2s_3^2s_2s_4s_2s_4s_3s_2^2s_3s_2^2s_3^2s_2^2s_4s_3s_2^2s_4s_3s_4s_2^2s_4\\
&\cdot s_2^2s_3^2s_4s_3s_2^2s_3s_2s_3s_4s_2^2s_3^2s_2^2s_4s_3s_2^2s_3s_2^2s_4s_2s_3s_2s_3s_2^2s_4s_3s_4s_2^2s_4s_2^2s_3^2s_4s_3s_4\\
&\cdot s_2s_4s_3s_2^2s_3s_2^2s_3^2s_2^2s_4s_3s_2^2s_4s_3s_4s_2^2s_4s_2^2s_3^2s_4s_3s_2.
\end{aligned}$$

By statement (γ) G has a base and strong generating system. Therefore we can apply a membership test. Its application yields the following equations completing the proof of (δ):

$$\begin{aligned}
a &= s_3c^3s_2c^{10}s_2c^6s_2c^8s_2c^3s_2c^5s_2c^{10}s_2c^2x_{11}c^9s_2^2c^{12}s_2c^2s_2c^8s_2c^9s_2c^{10}s_2c^3s_2c,\\
b_M &= s_3^2c^3s_2^2c^6s_2c^{12}s_2c^7s_2^2c^{11}s_2c^4s_2c^{12}x_{11}c^{10}s_2c^{10}s_2c^9s_2^2c^{12}s_2c^2s_2c^5,\\
x &= s_3c^8s_2c^{12}s_2c^2s_2c^7s_2c^{10}s_2c^3s_2x_{11}c^6s_2c^9s_2c^7s_2c^5s_2c^9s_2c^5s_2^2c^7,\\
y &= s_3s_4c^{11}s_2c^{11}s_2c^6s_2c^8s_2cs_2c^{12}s_2c^8s_2c^3s_2c^{11}x_{11}c^7s_2c^6s_2c^6s_2c^7s_2c^8s_2c^6\\
&\quad\cdot s_2c^4s_2c^2.
\end{aligned}$$

In order to show that the subgroup T equals the stabilizer S of the permutation representation U^G we now apply Gollan's double coset trick [6]. Its hypothesis has been satisfied as follows.

Using a computer it has been checked that the 2-point stabilizers

$$\mathrm{Stab}_T\left(1^{r_i}\right) = T \cap T^{r_i} = \langle y_{i1}, y_{i2}\rangle, \quad 1 \leqslant i \leqslant 11,$$

can be generated by the following words in the strong generators s_2, s_3 and s_4 in Table 4.3.

With these generators the first author checked in that $[\mathrm{Stab}_T(1^{r_i})]^{r_i^{-1}} \leqslant T$ for $i = 1, \dots, 11$. The proof was done by finding explicit words w in the generators of T that satisfy $(y_{i,j})^{r_i^{-1}} = w$. They are given in the following list:

$$\begin{aligned}
y_{1,1}^{(r_1^{-1})} &= b_T,\\
y_{1,2}^{(r_1^{-1})} &= c,\\
y_{2,1}^{(r_2^{-1})} &= y_{11,1}^2c^{12}b_Tc^5b_Tc^{10}b_Tc^6b_Tc^8b_Tc^7b_Tc^6,
\end{aligned}$$

Table 4.3

Generator		$o(y_{i,j})$
$y_{1,1}$	b_T	3
$y_{1,2}$	c	13
$y_{2,1}$	$c^5b_Tc^6b_Tc^5(c^4b_Tc^8b_Tc^6)^{-1}$	8
$y_{2,2}$	$c^{12}b_Tc^3b_Tc^{12}(c^7b_Tc^3)^{-1}$	12
$y_{3,1}$	$c^{12}b_Tc^6b_Tc^{10}(c^{10}b_Tc^5b_Tc^9)^{-1}$	18
$y_{3,2}$	$c^{10}b_Tc^9b_Tc^{11}b_Tc^5(c^9b_Tc^{12}b_Tc^9)^{-1}$	18
$y_{4,1}$	$cb_Tc^4b_Tcb_Tc^{10}(c^2b_Tc^5b_Tc^7b_Tc^4)^{-1}$	6
$y_{4,2}$	$c^6b_Tc^4b_Tc^{11}b_Tc^9(b_Tc^{12}b_Tc^5)^{-1}$	3
$y_{5,1}$	$c^7b_Tc^{11}b_Tc^8b_Tc^8(c^2b_Tc^5b_Tc^4)^{-1}$	6
$y_{5,2}$	$c^6b_Tc^{10}b_Tc^4b_Tc^6(c^3b_Tc^9b_Tc^3b_Tc^9)^{-1}$	6
$y_{6,1}$	$c^7b_Tc^6b_Tc^{11}b_Tc^4b_Tc^4(c^{12}b_Tc^4b_T^2c)^{-1}$	6
$y_{6,2}$	$c^9b_Tc^{10}b_Tc^6b_T^2c^{10}(b_Tc^2b_T^2c^6)^{-1}$	6
$y_{7,1}$	$c^9b_Tc^8b_Tc^6b_Tc^6(c^4b_Tc^4b_Tc^{11}b_Tc^2)^{-1}$	6
$y_{7,2}$	$c^{10}b_Tc^{11}b_Tc^{10}b_Tc^7(c^{11}b_Tc^9b_Tc^2)^{-1}$	6
$y_{8,1}$	$c^{12}b_T^2c^3b_Tc^7(c^4b_Tc^4b_Tc^4)^{-1}$	3
$y_{8,2}$	$c^{10}b_Tc^{11}b_Tc^{10}b_Tc^4b_Tc^3(cb_Tc^3b_Tc^4)^{-1}$	6
$y_{9,1}$	$c^7b_Tc^{10}b_Tc^6b_Tc^7(c^8b_Tcb_T^2c^{10})^{-1}c^4b_Tc^6b_Tc^2b_Tc^6(c^{10}b_T^2c^6b_Tc^2)^{-1}$	3
$y_{9,2}$	$c^2b_Tc^4b_Tcb_Tc^7b_Tc^{10}(b_Tc^4b_Tc^8b_Tc^9)^{-1}$	6
$y_{10,1}$	$cb_Tc^9b_Tc^{12}b_Tc^{11}(c^7b_Tc^5b_T^2c^3)^{-1}$	3
$y_{10,2}$	$c^{10}b_Tc^3b_Tc^9b_Tc^7b_T(c^{10}b_Tc^4b_Tc^3b_Tc^4)^{-1}$	6
$y_{11,1}$	$c^3b_Tc^4b_Tcb_Tc^{10}(c^{11}b_Tc^7b_Tcb_Tc^{12})^{-1}$	3
$y_{11,2}$	$c^2b_Tc^7b_T^2c^3b_Tc^3(b_Tc^6b_Tc^{10}b_Tcb_Tc^7)^{-1}$	2

$$y_{2,2}^{(r_2^{-1})} = y_{11,1}^2 b_Tc^6b_Tcb_Tc^6b_Tc^4b_T^2c^6b_Tc^5b_Tc^{11},$$

$$y_{3,1}^{(r_3^{-1})} = y_{11,1}y_{11,2}c^2b_Tc^9b_Tc^7b_Tc^4b_Tc^5b_Tc^2b_Tc^{10}b_Tc^3,$$

$$y_{3,2}^{(r_3^{-1})} = y_{11,2}y_{11,1}c^6b_Tc^4b_Tc^7b_Tc^2b_Tc^{11}b_Tc^{12}b_Tc^5b_Tcb_Tc^{10},$$

$$y_{4,1}^{(r_4^{-1})} = y_{11,1}c^{11}b_Tc^5b_Tc^9b_Tc^2b_Tc^6b_Tcb_Tc^3b_Tc^2b_Tc_T^bc^8,$$

$$y_{4,2}^{(r_4^{-1})} = y_{11,1}c^8b_Tc^2b_Tc^6b_Tc^6b_Tc^9b_Tc^{12}b_Tc^{11}b_Tcb_Tc^8,$$

$$y_{5,1}^{(r_5^{-1})} = y_{11,1}y_{11,2}c^8b_Tc^4b_Tc^{12}b_Tc^7b_Tc^7b_Tc^3b_Tc^6b_T,$$

$$y_{5,2}^{(r_5^{-1})} = y_{11,1}c^2b_Tc^{11}b_T^2c^2b_Tc^6b_Tc^9b_Tc^6b_Tc^9b_Tc^{11}b_Tc^3,$$

$$y_{6,1}^{(r_6^{-1})} = c^{12}b_Tcb_Tc^{12}b_T^2c^6b_Tc^2b_Tc^5,$$

$$y_{6,2}^{(r_6^{-1})} = y_{11,1}c^6 b_T c^5 b_T c^8 b_T^2 c^5 b_T c^7 b_T c^{10} b_T c^5,$$

$$y_{7,1}^{(r_7^{-1})} = y_{11,1}^2 b_T c^{10} b_T c^5 b_T c^{12} b_T c^7 b_T c^{10} b_T c^5,$$

$$y_{7,2}^{(r_7^{-1})} = y_{11,1}^2 b_T c^5 b_T c^9 b_T c^8 b_T c^8 b_T c^{10} b_T c^2 b_T c^{11} b_T c^{10},$$

$$y_{8,1}^{(r_8^{-1})} = y_{11,2}c^7 b_T c^7 b_T c^6 b_T c^7 b_T c^{12} b_T^2 c b_T c^{11},$$

$$y_{8,2}^{(r_8^{-1})} = y_{11,1} b_T c^3 b_T c^6 b_T c^2 b_T c^{11} b_T c^8 b_T c^5 b_T c^5 b_T c^8,$$

$$y_{9,1}^{(r_9^{-1})} = y_{11,2} b_T c^3 b_T c^6 b_T c^2 b_T c^{12} b_T c^5 b_T^2 c^7 b_T c^9,$$

$$y_{9,2}^{(r_9^{-1})} = y_{11,1} y_{11,2} c^3 b_T c^7 b_T c^6 b_T c^7 b_T c^{10} b_T^2 c^3,$$

$$y_{10,1}^{(r_{10}^{-1})} = y_{11,2} c^7 b_T c^6 b_T c^9 b_T c^{10} b_T c^{12} b_T c^5 b_T c^9,$$

$$y_{10,2}^{(r_{10}^{-1})} = y_{11,1} y_{11,2} c b_T c^9 b_T c b_T^2 c^6 b_T c^{12} b_T c^6 b_T c^6 b_T c^3,$$

$$y_{11,1}^{(r_{11}^{-1})} = c^4 b_T c b_T c^{10} b_T c^{10} b_T c^2 b_T c^9 b_T c^{12},$$

$$y_{11,2}^{(r_{11}^{-1})} = y_{11,2} c^6 b_T c^{10} b_T c^{11} b_T^2 c^9 b_T c^3 b_T c^9.$$

Thus condition (a) of Gollan's result holds.

Let $K = T \cap T^{b^{-1}}$. As K is a subgroup of T, each T-orbit r_i^T decomposes into s_i K-orbits $(r_i k_{i,j})^K$ for $i = 1, 2, \ldots, 11$. The numbers s_i and the $T-K$ double coset representatives $k_{i,j}$ were determined computationally by the first author using the large permutation representation Ω. He showed that K decomposes the T-orbits of Ω as shown in Table 4.4.

All together there are 4188 double coset representatives $k_{i,j}$ defining the starting points $r_i k_{i,j}$ of the K-orbits $(r_i k_{i,j})^K$. They cannot be listed here. Condition (b) of Gollan's double coset trick [6] requires to check that

$$r_i k_{i,j} r_j^{-1} \in \bigcup_{i=1}^{11} T r_i T \quad \text{for all } 1 \leqslant i \leqslant 11 \text{ and } 1 \leqslant j \leqslant t_i.$$

With the explicit elements $k_{i,j}$ defined above these 4188 conditions were checked by the first author using the strong base and generating system given above. The resulting relations

Table 4.4

	r_1^T	r_2^T	r_3^T	r_4^T	r_5^T	r_6^T	r_7^T	r_8^T	r_9^T	r_{10}^T	r_{11}^T
# of K-orbits	1	12	9	13	53	97	158	253	259	358	2975

$r_i k_{i,j} r_j^{-1} = w$ with words w in the strong generators s_1, s_2, s_3 and s_4 of G can be received from the first author by e-mail. As an example we state that:

$$\begin{aligned} k_{11,1288} &:= y_{2,2}^3 y_{2,1} y_{2,2}^2 y_{2,1} y_{2,2}^3 b_T, \\ x_{11} k_{11,1288} x_2^{-1} &= y_{11,1} c^2 b_T c b_T^2 c^3 b_T^2 c^2 b_T^2 c^5 b_T c b_T c^2 b_T c^2 b_T x_{11} y_{2,1} y_{2,2}^2 y_{2,1} y_{2,2}^2 \\ &\quad \cdot y_{2,1} y_{2,2}^3 y_{2,1}^2 y_{2,2}^2 b_T y_{2,1}^2 y_{2,2} y_{2,1} y_{2,2} y_{2,1} y_{2,2} y_{2,1}^2 y_{2,2}^3 y_{2,1} y_{2,2}. \end{aligned}$$

The 4188 relations hold also for the corresponding matrices in the group $G = \langle x, y, d, a \rangle \leqslant \mathrm{GL}_{248}(11)$. Now Gollan's result asserts that the stabilizer $S = T$. Hence $|G| = |T| \cdot 14{,}312{,}700 = 2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$.

By Table 9.1 the element $z = x^8$ is a 2-central involution of G. The group $G = \langle H, E \rangle$ is perfect, because H and E are perfect by Lemmas 2.2 and 3.1, respectively. Now the odd order theorem of Feit and Thompson implies that any proper normal subgroup N of G has even index. Suppose that $N \cap H \neq 1$. By the proof of Lemma 3.1 the group G has a unique conjugacy class of involutions. Thus $z \in N$. Hence $A \leqslant N$ because $E/A \cong L_5(2)$ acts irreducibly on A by Lemma 3.1. Since A is a normal subgroup in a Sylow 2-subgroup of G by Lemma 2.2, it follows that $|G : N|$ is odd, a contradiction. Thus $N \cap H = 1$, and $|N|$ is odd. As G has a unique conjugacy class of involutions it follows that $C_N(w) = N \cap C_G(w) \leqslant O(C_G(w)) = 1$ for all involutions w of G. Now the theorem of Brauer and Wielandt implies that $N = 1$.

Using the faithful permutation representation of T afforded by its double coset Tr_2T of degree 17,199 and Magma the first author has checked that T has two conjugacy classes of involutions y_1^T and y_2^T with centralizer orders $|C_T(y_1)| = 2^{12} \cdot 3^{10} \cdot 7$ and $|C_T(y_2)| = 2^{10} \cdot 3^2$, respectively. Since there is only one conjugacy class of involutions in G, both involutions y_1 and y_2 fuse to z in G. In order to determine the order $|C_G(z)|$ one calculates the number of fixed points of z on the permutation representation module $\Omega = U^G$ constructed above. It follows that

$$10{,}200 = \pi(z) = \left|C_G(z)\right| \left[\frac{1}{|C_T(y_1)|} + \frac{1}{|C_T(y_2)|} \left|C_G(z)\right| \left(\frac{1}{774144} + \frac{1}{9216} \right) \right]$$

by [8, p. 64]. Hence $|C_G(z)| = 2^{15} \cdot 3^4 \cdot 5 \cdot 7 = |H|$, and $C_G(z) \cong H$. Thus $G = \mathfrak{G}$ is a simple group of Th-type. □

Remark 4.3. Table 4.5 lists computational resources used at the Computer Center of the University of Karlsruhe and the Institute of Experimental Mathematics for the construction permutation group of degree 143,127,000 corresponding to the simple matrix group $\mathfrak{G}$.

For details concerning the implementation and the algorithms used in these demanding calculations see Weller's articles [21,22].

Table 4.5

Task	Architecture	Nodes	Elapsed time	Parallel (%)
Sequential setup of hash table	SP-256	1	2:54	–
Hash table of 143 mil. spaces	SP-256	64	99:06	98
Compute b on 143 mil. points	SP-256	64	133:32	98
Compute b_T on 143 mil. points	SP-256	64	133:05	98
Construct $y_{i,j}$	RS-590	1	3:34	–
Construct D_2 and D_3	RS-590	1	61:16	–
Construct D_5	RS-590 cluster	4	366:18	100

5. Determination of 3-singular conjugacy classes

In this section it is proved that all finite simple groups G of Th-type have isomorphic Sylow 3-subgroups S. This is done by showing that S has a noncyclic subgroup D_1 of order 9 such that $M = C_G(D_1)$ is a subgroup of order 3^9, and $N_G(M)/M \cong \mathrm{GL}_2(3)$. Then it is shown that any simple group G of Th-type has 3 conjugacy classes of elements of order 3 represented by explicitly given elements d_0, d_2, and d_4 in $H = C_G(z)$, and that their normalizers $N_G(d_i)$ are uniquely determined by d_i up to isomorphism. For each of these normalizers $N_G(d_i)$ a system of generators and defining relations and a faithful permutation representation are given. Furthermore, their character tables and systems of representatives of their conjugacy classes are calculated. Thus a complete classification of all 3-singular conjugacy classes of G has been obtained.

Lemma 5.1. *Let G be a finite simple group of* Th*-type having an involution z with centralizer*

$$H = C_G(z) = \langle z, c, d, e, s, t, u\rangle = \langle u_i, v_j, x, y, d \mid 1 \leqslant i \leqslant 5,\ 1 \leqslant j \leqslant 4\rangle$$

defined in Lemma 2.2. *In H define the following elements*:

$$\begin{aligned}
d_0 &= x^8,\\
d_1 &= t^{-1}s^{-1}t^{-1}eus^{-1}tdusedut,\\
d_2 &= sesus^2zdcdedusedt,\\
d_3 &= t^{-1}esustzduds^{-1}edsu^{-1},\\
k_1 &= xv_2x^{-1},\\
k_2 &= v_2,\\
k_3 &= u_2u_4v_4v_1^{-1},\\
k_4 &= y^{-1}v_3^{-1}v_1^{-1}v_3^{-1}u_2^{-1}x^{-1}u_2^{-1}u_3^{-1}v_4^{-1}u_3xv_3y,\\
k_5 &= t^{-1}s^{-1}us^{-1}dudes^{-1}ut,\\
k_6 &= st^{-1}usezdu^{-1}dsuts,
\end{aligned}$$

$$\begin{aligned}
k_7 &= k_5 d_2 d_1^2,\\
k_8 &= s^{-1} esdudcdu^{-1}d,\\
k_9 &= s^2 eu^2 duds^{-1}t^{-1}u^{-1}s^{-1}t^{-1},\\
k_{10} &= esudcdu^{-1}dsudet.
\end{aligned}$$

Then the elements d_i, $0 \leqslant i \leqslant 3$, k_j, $1 \leqslant j \leqslant< 4$, k_l, $5 \leqslant l \leqslant 8$, *have orders* 3, 4 *and* 2, *respectively. Furthermore,* k_9 *and* k_{10} *have orders* 8 *and* 4, *respectively.*

With these notations the following statements hold:

(a) $H_3 = \langle d_i \mid 0 \leqslant i \leqslant 3\rangle \leqslant H$ *is a Sylow* 3*-subgroup of* H *with center* $Z(H_3) = \langle d_2\rangle$, *and* $C = \langle d_0, d_1, d_2\rangle$ *is a maximal elementary abelian normal subgroup of* H_3, *which is a wreathed product of a cyclic group of order* 3 *by a cyclic group of order* 3.

(b) d_0, d_2, *and* $d_4 = d_1d_2$ *form a complete set of representatives of the conjugacy classes of elements of order* 3 *in* H.

(c) d_1, d_2, *and* d_3 *are conjugate in* H.

(d) $C_0 = C_H(d_0) = \langle d_0\rangle \times H_0$, *where* $H_0 = \langle H_0', k_5\rangle$, *and the commutator subgroup* H_0' *of* H_0 *is the central product* $L_1 * L_2$ *with amalgamated subgroup* $\langle z\rangle$ *of its normal subgroups* $L_1 = \langle k_1, k_2, d_1\rangle$ *and* $L_2 = \langle k_3, k_4, d_2\rangle$ *which are both isomorphic to* $\mathsf{SL}_2(3)$.

(e) $N_H(d_0) = (\langle d_0\rangle \times H_0) : \langle k_6\rangle$, *and* $d_0^{k_6} = d_0^2$.

(f) $C_2 = C_H(d_2) = C : H_2$, *where* $C = \langle d_0, d_1, d_2\rangle$ *and* $H_2 = \langle k_1, k_2, d_3\rangle \cong \mathsf{SL}_2(3)$.

(g) $T_2 = \langle k_1, k_2\rangle$ *is a quaternion group of order* 8 *with center* $\langle z\rangle$; *it is a Sylow* 2*-subgroup of* H_2, *and* $C = O(C_2)$ *is elementary abelian of order* 27.

(h) $N_H(d_2) = \langle C_2, k_7\rangle$, *and* $d_2^{k_7} = d_2^2$.

(i) $C_4 = C_H(d_4) = \langle d_4\rangle \times H_4$, *where* $H_4 = \langle d_0, d_1d_2^{-1}, k_9, k_{10}\rangle \cong 2A_6$ *has a Sylow* 2*-subgroup* $T_4 = \langle k_9, k_{10}\rangle$ *which is a generalized quaternion group of order* 16.

(j) $N_H(d_4) = \langle C_4, k_7\rangle$, $d_4^{k_7} = d_4^2$, $|C_H(k_7)| = 2^{10}\cdot 3$, $|C_H(d_4) \cap C_H(k_7)| = 2^3 \cdot 3$, *and* k_7 *and* k_7z *are conjugate in* $N_H(d_4)$.

(k) *The* 3*-elements* d_0, d_2 *and* d_4 *are not conjugate in any simple group* G *of* Th*-type.*

Proof. The statements (a)–(j) have been checked by means of MAGMA and the faithful permutation representation of H given in Lemma 2.2.

(k) $C_G(d_0)$, $C_G(d_2)$ and $C_G(d_4)$ have Sylow 2-subgroups of order 32, 8 and 16 by (d), (f) and (i), respectively. Thus (k) holds. □

Lemma 5.2. *Keep the notations of Lemma* 5.1. *Let* $d_5 = d_1d_2^{-1}$, *and let* A *be the unique maximal elementary abelian normal subgroup of the Sylow* 2*-subgroup* S *of* H *defined in Lemma* 2.2. *Let* $E = N_G(A)$ *and* $A_0 = \langle z, k_1k_4, k_1k_2k_3^{-1}\rangle \leqslant H_0$. *Then the following statements hold*:

(a) A_0 *is a maximal elementary abelian normal subgroup of the Sylow* 2*-subgroup* $S_0 = \langle k_i \mid 1 \leqslant i \leqslant 5\rangle$ *of* H_0, *and* $|A_0| = 8$.

(b) $D_0 = N_{H_0}(A_0) = E \cap H_0 = \langle S_0, d_5\rangle$, *and* $C_E(d_0) = \langle d_0\rangle \times \langle D_0, t_0\rangle$, *where*

$$
\begin{aligned}
t_0 := {} & v_3 y v_2 a v_3^2 a^{-1} y^{-1} u_3^{-1} v_2 v_4^{-1} v_2^{-1} u_1 v_3 y v_2 a v_3^2 y x v_2 y^{-1} a^{-1} \\
& \cdot y^{-1} x^2 a x u_1 v_3 y v_2 a v_3^2 y x v_2 y^{-1} a^{-1} y^{-1}
\end{aligned}
$$

has order 7.

(c) $E_0 = \langle D_0, t_0 \rangle = \langle k_i, d_5, t_0 \mid 1 \leqslant i \leqslant 5 \rangle$ *has the following set* $\mathcal{R}(E_0)$ *of defining relations*:

$$
\begin{aligned}
& k_1^4 = k_2^4 = k_3^4 = k_4^4 = k_5^2 = d_5^3 = t_0^7 = 1, \\
& k_2^{-1} k_1^2 k_2^{-1} = k_3^{-1} k_1^2 k_3^{-1} = k_4^{-1} k_1^2 k_4^{-1} = k_1^{-1} k_2^{-1} k_1 k_2^{-1} = 1, \\
& [k_1, k_3] = [k_2, k_3] = [k_1, k_4] = [k_2, k_4] = 1, \\
& k_3^{-1} k_4^{-1} k_3 k_4^{-1} = k_1^{-1} k_5 k_2^{-1} k_5 = \bigl(k_3^{-1} k_5\bigr)^2 = 1, \\
& k_2^{-1} d_5^{-1} k_1 d_5 = k_4^{-1} d_5^{-1} k_3 d_5 = \bigl(k_5 d_5^{-1}\bigr)^2 = 1, \\
& d_5^{-1} k_2^{-1} k_1^{-1} d_5 k_1 = k_1^{-1} t_0 k_1^2 t_0^{-1} k_4^{-1} = 1, \\
& k_5 k_4^{-1} k_3^{-1} k_5 k_4 = \bigl(k_2 t_0^{-1} k_5\bigr)^2 = 1, \\
& t_0 k_3^{-1} d_5^{-1} k_2^{-1} t_0 k_5 = k_1^{-1} t_0 d_5^{-1} k_4^{-1} t_0 d_5^{-1} = 1, \\
& t_0^2 d_5 k_3 t_0 k_3^{-1} = 1.
\end{aligned}
$$

(d) $W = \langle K_0, E_0 \rangle \leqslant G$ *is a simple subgroup of* G *with a unique conjugacy class* z^W *of involutions, and* $C_W(z) = H_0$.

(e) $C_G(d_0) = \langle d_0 \rangle \times W$, *and* $W \cong G_2(3)$.

(f) *The involution* k_6 *defined in Lemma* 5.1 *is the unique involution of* H *satisfying the following conditions*:

$$
\begin{aligned}
& k_6 \in D \cap N_H(A_0) \cap N_H(d_0), \quad [k_6, d_5] = [k_6, s_0] = 1, \\
& [k_6, a] = 1 \quad \text{for all } a \in A_0, \quad \text{and} \quad d_0^{k_6} = d_0^2,
\end{aligned}
$$

where $s_0 = k_5 k_4 t_0 d_5^2$.

(g) $|N_G(W) : W C_G(W)| = 2$.

(h) $N_0 = N_G(d_0) = \langle k_i, d_0, d_1, d_2, s_0 \mid 1 \leqslant i \leqslant 6 \rangle = (\langle d_0 \rangle \times W) : \langle k_6 \rangle$ *has the following set* $\mathcal{R}(N_0)$ *of defining relations*:

$$
\begin{aligned}
& k_1^4 = k_2^4 = k_3^4 = k_4^4 = k_5^2 = k_6^2 = d_0^3 = d_1^3 = d_2^3 = s_0^7 = 1, \\
& k_2^{-1} k_1^2 k_2^{-1} = k_3^{-1} k_1^2 k_3^{-1} = k_4^{-1} k_1^2 k_4^{-1} = k_1^{-1} k_2^{-1} k_1 k_2^{-1} = 1, \\
& [k_1, k_3] = [k_2, k_3] = [k_1, k_4] = [k_2, k_4] = [k_3, d_1] = [k_4, d_1] = 1, \\
& k_3^{-1} k_4^{-1} k_3 k_4^{-1} = k_1^{-1} k_5 k_2^{-1} k_5 = \bigl(k_3^{-1} k_5\bigr)^2 = k_2^{-1} d_1^{-1} k_1 d_1 = 1, \\
& \bigl(k_5 d_1^{-1}\bigr)^2 = [k_1, d_2], \qquad [k_2, d_2] = k_3^{-1} d_2^{-1} k_4 d_2 = \bigl(k_5 d_2^{-1}\bigr)^2 = 1, \\
& [d_1, d_2] = d_1^{-1} k_2^{-1} k_1^{-1} d_1 k_1 = k_5 k_4^{-1} k_3^{-1} k_5 k_4 = 1,
\end{aligned}
$$

$$\begin{gathered}
k_3^{-1}s_0^{-1}k_3^{-1}k_1s_0k_5 = s_0k_5k_4^{-1}k_2^{-1}s_0^{-1}k_2^{-1} = s_0^2k_5k_4^{-1}s_0k_2 = 1,\\
k_1^{-1}s_0^{-1}k_4^{-1}k_1^{-1}s_0k_4 = d_2d_1^{-1}s_0k_1^{-1}s_0^{-1}k_3s_0 = 1,\\
d_1^{-1}s_0^{-1}d_1^{-1}s_0d_1^{-1}s_0^{-1}d_1s_0d_1s_0^{-1}d_1s_0 = \big(d_1s_0^{-1}\big)^6 = 1,\\
k_1^{k_6} = k_4^{-1}, \qquad k_2^{k_6} = k_3k_4, \qquad k_3^{k_6} = k_1k_2^{-1}, \qquad k_4^{k_6} = k_1^{-1},\\
k_5^{k_6} = k_1^2k_5, \qquad d_1^{k_6} = d_2^{-1}, \qquad d_2^{k_6} = d_1^{-1}, \qquad [k_6, s_0] = 1, \qquad d_0^{k_5} = d_0^2,\\
[d_0, k_i] = 1 \quad \textit{for all } 1 \leqslant i \leqslant 5, \qquad [d_0, d_1] = [d_0, d_2] = [d_0, s_0] = 1.
\end{gathered}$$

(i) *Representatives of the* 51 *conjugacy classes* $n_i^{N_0}$ *of* $N_0 = N_G(d_0)$ *and the corresponding centralizer orders* $|C_{N_0}(n_i)|$ *are given in Table* 9.6.

(j) *The character table of* N_0 *is given in Table* 10.6.

(k) N_0 *has a faithful permutation representation of degree* 44,226 *with stabilizer* $H_0 = \langle k_i, d_1, d_2 \mid 1 \leqslant i \leqslant 5\rangle$.

Proof. Assertions (a) and (b) have been checked computationally by means of MAGMA and the faithful permutation representations of H and E given in Lemmas 2.2 and 3.1, respectively. Furthermore, the character tables of H_0 and E_0 have been calculated by means of MAGMA. They are stated in Section 10.

(c) It also has been verified computationally that $\mathcal{R}(E_0)$ is a set of defining relations for $E_0 = \langle k_i, d_5, s_0 \mid 1 \leqslant i \leqslant 5\rangle$, and that A_0 is normal in E_0 with $e_0/A_0 \cong L_3(2)$.

(d) Lemma 5.1 states that $C_H(d_0) = \langle d_0\rangle \times H_0$. Let $W = \langle H_0, E_0\rangle \leqslant G$, where G is a simple group of Th-type having an involution z such that $H = C_G(z)$. By Lemma 5.1 we know that H_0 has center $Z(H_0) = \langle z\rangle$. Hence $H_0 \leqslant C_W(z)$. Let $w \in C_W(z)$. Then $w \in C_G(z) = H$, and $w \in C_G(d_0)$, because $W = \langle H_0, E_0\rangle \leqslant C_G(d_0)$. Thus $w \in H \cap C_G(d_0) = C_H(d_0) = \langle d_0\rangle \times H_0$, and $w \in H_0$. Therefore $C_W(z) = H_0$.

Furthermore, S_0 is a Sylow 2-subgroup of W. Using the faithful permutation representation of $E = N_G(A)$ given in Lemma 3.1 and MAGMA it has been verified that E_0 has 2 conjugacy classes of involutions. They are represented by z and k_5.

Suppose that T is a normal subgroup of W with $|W : T| = 2$. Since $E_0/A_0 \cong L_3(2)$, and $L_3(2)$ operates irreducibly on A_0, the group E_0 is perfect. Thus $|E_0T/T| = |E_0/T \cap E_0| = 1$ because E_0T/T is a subgroup of W/T. Hence $S_0 \leqslant E_0 \leqslant T$, and $|W : T|$ is odd, a contradiction. Therefore W does not have any normal subgroups of index 2.

Since $k_5 \in S_0 - S_0'$ Thompson's transfer lemma now implies that k_5 is conjugate to z in W. In particular, W has a unique conjugacy class of involutions.

Suppose that $X = O(W) \neq 1$. Let $V = \langle z, k_5\rangle$ operate on X by conjugation. Then

$$|X|\big|C_X(V)\big|^2 = \big|C_X(z)\big|\big|C_X(k_5)\big|\big|C_X(k_5z)\big| = \big|C_X(z)\big|^3$$

by the Brauer–Wielandt theorem. From the character table of H_0 follows that $|C_{H_0}(k_5)| = 2^3$. Hence $|C_X(V)| = 1$. Furthermore, $C_X(z) = C_W(z) \cap X = H_0 \cap X \leqslant O(H_0) = 1$, because $H_0' = \mathsf{SL}_2(3) * \mathsf{SL}_2(3)$ by Lemma 5.1. Therefore, $X = 1$.

Let $Y \neq 1$ be a minimal normal subgroup of W. Then $O(W) = 1$ implies that $O(Y)$ is a characteristic subgroup of Y. Hence Y has even order, and all involutions of W belong to Y.

Therefore W/Y has odd order. Since E_0 and H_0 do not have proper normal subgroups of odd index, it follows that $W = \langle H_0, E_0\rangle \leqslant Y \leqslant W$. Thus $W = Y$, and W is a simple group with a unique conjugacy class of involutions.

Using algorithm [12] and MAGMA we find the missing relations between the additional generator s_0 of $W = \langle H_0, E_0\rangle = \langle H_0, S_0\rangle$ and the generators of H_0 as follows.

By (a) A_0 is a maximal elementary abelian normal subgroup of the Sylow 2-subgroup S_0 of H_0. From (c) and (d) follows that $E_0 = N_W(A_0)$. The character tables of H_0, E_0 and $D_0 = N_{H_0}(A_0)$ are stated in Section 10. Using their notations it has been checked by means of MAGMA and Kratzer's algorithm [10] that there is a unique compatible pair

$$(\chi, \tau) = (\chi_7 + \chi_9 + \chi_{18}, \tau_9) \in mf\,\mathrm{char}(H_0) \times mf\,\mathrm{char}(E_0)$$

with common restriction to D_0:

$$\chi|_{D_0} = \tau|_{D_0} = \delta_3 + \delta_4 + \delta_9, \quad \text{where } \delta_k \in \mathrm{Irr}_{\mathbb{C}}(D_0),$$

which by means of algorithm [12] leads to a simple matrix group $\mathfrak{W} \leqslant \mathrm{GL}_{14}(5)$ with a centralizer $\mathfrak{C}_{\mathfrak{W}}(z) \cong H_0$ of an involution $\mathfrak{z}$ corresponding to the involution z of H_0.

By the character tables of H_0 and E_0 it follows that the prime field $F = \mathrm{GF}(5)$ is a splitting field for the irreducible FH_0- and FE_0-modules corresponding to the characters χ_7, χ_9, χ_{18} of H_0 and τ_9 of E_0, respectively, because 5 does not divide $|H_0||E_0|$. Using the faithful permutation representations of H_0 and E_0 with degrees 32 and 168, respectively, the simple FH_0- and FE_0-modules of degrees 3, 3, 8 and 14 corresponding to χ_7, χ_9, χ_{18} and τ_9, respectively, have been constructed.

(e) From (d) and the main theorem of Janko's article [9] follows that $W \cong G_2(3)$, where the finite group of Lie type $G_2(3)$ and its properties are described in R.W. Carter's book [3, p. 206].

(f) Assertion (f) has been verified computationally by means of MAGMA and the faithful permutation representation of $E = N_G(A)$ described in Lemma 3.1.

(g) By (d) and (e) we know that W is a simple group which is isomorphic to the Chevalley group $G_2(3)$. Theorem 12.5.1 of [3, p. 211] states that the outer automorphism group $\mathrm{Out}(G_2(3)) \cong \mathbb{Z}/2\mathbb{Z}$. Since $k_6 \in N_G(W) - C_G(W)W$ by (f) it follows that $N_G(W)/C_G(W)W \cong \mathbb{Z}/2\mathbb{Z}$.

(h) As $H_0 = \langle k_i, d_1, d_2 \mid 1 \leqslant i \leqslant 5\rangle = \langle H_0', k_5\rangle$, and $H_0' \cong \mathsf{SL}_2(3) * \mathsf{SL}_2(3)$ it is easy to determine a defining set $\mathcal{R}(H_0)$ of relations for H_0 with respect to its given generators. In order to find a defining set $\mathcal{R}(W)$ of defining relations for the simple group W we apply the main theorem of Janko's article [9] again, which asserts that W is uniquely determined up to isomorphism by the centralizer $C_W(z) = H_0$ of its involution z. It allows us to construct W from $C_W(z)H_0$ by means of algorithm [12].

Since $3_1 \oplus 3_2 \oplus 8|_{D_0} = 14|_{D_0}$ the Meat-Axe isomorphism test has been used to identify the two 14-dimensional FD_0-module by means of a transformation matrix. It follows that $\mathfrak{W} \leqslant \mathrm{GL}_{14}(5)$ is generated by the following matrices:

$$(k_1)=\begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&4&2&2&0&0&0&0&0&0&0&0\\
0&0&0&0&3&4&0&0&0&0&0&0&0&0\\
0&0&0&0&3&2&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&4&4&3&1&3&0&2\\
0&0&0&0&0&0&2&2&4&4&2&1&2&3\\
0&0&0&0&0&0&3&1&3&1&1&0&3&3\\
0&0&0&0&0&0&1&1&1&4&4&0&1&2\\
0&0&0&0&0&0&4&2&4&3&2&2&2&3\\
0&0&0&0&0&0&4&4&3&3&1&4&1&3\\
0&0&0&0&0&0&0&4&2&0&2&4&1&1\\
0&0&0&0&0&0&2&1&1&4&0&2&1&3
\end{pmatrix},$$

$$(k_2)=\begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&3&2&0&0&0&0&0&0&0&0\\
0&0&0&1&2&2&0&0&0&0&0&0&0&0\\
0&0&0&4&2&2&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&2&3&2&3&2&0&1&3\\
0&0&0&0&0&0&1&3&0&3&1&1&0&0\\
0&0&0&0&0&0&2&2&2&4&0&1&2&0\\
0&0&0&0&0&0&2&2&0&1&1&1&0&3\\
0&0&0&0&0&0&0&1&4&4&1&4&0&2\\
0&0&0&0&0&0&4&0&4&1&1&0&4&1\\
0&0&0&0&0&0&4&3&1&0&2&1&0&3\\
0&0&0&0&0&0&1&0&1&2&2&1&3&1
\end{pmatrix},$$

$$(k_3)=\begin{pmatrix}
1&2&0&0&0&0&0&0&0&0&0&0&0&0\\
0&4&0&0&0&0&0&0&0&0&0&0&0&0\\
3&3&4&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&4&2&4&0&0&2&2\\
0&0&0&0&0&0&1&4&3&4&4&4&0&2\\
0&0&0&0&0&0&4&2&1&1&4&2&3&4\\
0&0&0&0&0&0&3&0&0&2&0&2&0&3\\
0&0&0&0&0&0&4&3&2&3&4&0&2&4\\
0&0&0&0&0&0&1&4&4&0&1&1&0&3\\
0&0&0&0&0&0&4&3&0&4&2&3&3&1\\
0&0&0&0&0&0&3&0&2&1&1&3&3&4
\end{pmatrix},$$

$$(k_4)=\begin{pmatrix}
4&3&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&4&4&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&3&3&3&3&1&3&0&1\\
0&0&0&0&0&0&2&4&4&0&4&0&4&3\\
0&0&0&0&0&0&0&0&3&4&1&2&2&1\\
0&0&0&0&0&0&2&0&0&3&1&0&2&1\\
0&0&0&0&0&0&3&4&2&3&1&3&0&2\\
0&0&0&0&0&0&2&3&3&1&4&0&1&4\\
0&0&0&0&0&0&3&2&4&1&4&2&3&0\\
0&0&0&0&0&0&4&1&4&4&1&2&3&3
\end{pmatrix},$$

$$(k_5)=\begin{pmatrix}
3&0&4&0&0&0&0&0&0&0&0&0&0&0\\
1&4&1&0&0&0&0&0&0&0&0&0&0&0\\
3&0&2&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&4&2&0&0&0&0&0&0&0&0\\
0&0&0&1&1&1&0&0&0&0&0&0&0&0\\
0&0&0&3&1&2&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&2&3&4&3&1&0&4&1\\
0&0&0&0&0&0&3&3&1&2&2&0&2&3\\
0&0&0&0&0&0&0&1&4&1&1&2&1&1\\
0&0&0&0&0&0&2&4&1&4&4&3&4&0\\
0&0&0&0&0&0&0&3&1&0&1&2&2&3\\
0&0&0&0&0&0&0&2&0&1&4&4&1&3\\
0&0&0&0&0&0&4&3&4&3&3&4&4&1\\
0&0&0&0&0&0&1&4&3&0&3&0&2&3
\end{pmatrix},$$

$$(d_1)=\begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&2&2&2&0&0&0&0&0&0&0&0\\
0&0&0&3&2&1&0&0&0&0&0&0&0&0\\
0&0&0&3&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&4&3&0&4&2&4&4&0\\
0&0&0&0&0&0&2&1&2&2&4&1&2&4\\
0&0&0&0&0&0&3&4&4&0&3&4&0&0\\
0&0&0&0&0&0&2&3&4&4&0&1&3&3\\
0&0&0&0&0&0&0&2&4&0&1&0&0&4\\
0&0&0&0&0&0&4&0&1&1&3&0&3&3\\
0&0&0&0&0&0&1&3&1&0&2&2&0&3\\
0&0&0&0&0&0&2&2&4&0&2&1&0&3
\end{pmatrix},$$

$$(d_2)=\begin{pmatrix}
3&0&4&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
3&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&2&1&3&4&2&0&0\\
0&0&0&0&0&0&3&1&0&1&1&1&3&2\\
0&0&0&0&0&0&0&4&4&0&2&0&3&0\\
0&0&0&0&0&0&3&4&0&1&1&3&4&2\\
0&0&0&0&0&0&0&1&1&2&3&2&4&1\\
0&0&0&0&0&0&4&1&2&1&2&0&3&3\\
0&0&0&0&0&0&1&0&1&4&1&3&1&4\\
0&0&0&0&0&0&3&3&1&2&2&3&3&2
\end{pmatrix},$$

$$(s_0)=\begin{pmatrix}
3&2&3&3&4&1&4&4&2&2&2&1&4&4\\
2&3&2&2&1&4&0&0&0&0&0&0&0&0\\
4&1&4&4&2&3&2&0&1&2&0&0&2&2\\
4&1&4&2&1&4&0&0&1&0&2&1&2&1\\
3&2&3&4&2&3&2&1&1&1&0&0&0&2\\
1&4&1&3&4&1&3&4&4&4&0&0&0&3\\
0&4&0&1&4&0&2&2&4&4&2&0&1&0\\
2&2&0&1&4&0&2&4&2&0&2&2&0&1\\
3&4&0&2&4&1&4&4&1&0&2&3&1&0\\
1&3&0&1&2&3&4&1&0&2&1&0&3&2\\
1&3&0&4&0&4&1&4&3&0&2&3&1&3\\
3&3&0&2&3&0&3&2&1&4&2&3&4&2\\
3&2&0&1&1&2&4&0&3&1&4&4&2&1\\
3&0&0&3&1&4&3&1&0&3&0&0&1&4
\end{pmatrix}.$$

Another application of MAGMA to the matrix group

$$\mathfrak{W}=\big\langle (k_i),(d_1),(d_2),(s_0)\mid 1\leqslant i\leqslant 5\big\rangle\leqslant \mathrm{GL}_{14}(5)$$

enables one to construct a faithful permutation representation of $\mathfrak{W}$ of degree 7371 with stabilizer

$$\mathfrak{H}_0=\big\langle (k_i),(d_1),(d_2)\mid 1\leqslant i\leqslant 5\big\rangle.$$

Let $(z)=(k_1)^2$. Then $\mathfrak{H}_0=\mathfrak{C}_{\mathfrak{W}}(\mathfrak{z})\cong H_0$. In particular, $\mathfrak{W}=W$ by Janko's theorem mentioned above.

Using now this faithful permutation representation and MAGMA one is able to determine a set $\mathcal{R}(W)$ of defining relations for W with respect to its generators. It consists of all the relations stated in assertion (h) which do not involve the additional generators d_0 and k_6 of $N_0=N_G(d_0)$. In particular, $\mathcal{R}(H_0)$ is a subset of $\mathcal{R}(W)$.

By (f) the involution k_6 normalizes $H_0=C_W(z)$. Hence it is very easy to determine x^{k_6} as a word in the generators k_1, k_2, k_3, k_4, k_5, d_1 and d_2 of H_0 for each generator x of

H_0 using MAGMA and the faithful permutation representation of H given in Lemma 2.2. The resulting relations involving $k_i^{k_6}$ and $d_j^{k_6}$ for $i = 1, 2, 3, 4, 5$ and $j = 1, 2$ are stated in the third and second last lines of (h). The remaining relations $[k_6, s_0] = 1$, $d_0^{k_6} = d_0^2$, and $[d_0, w] = 1$ for all $w \in W$ are restatements of some relations in (f) and (e), respectively.

(k) Using MAGMA and the permutation of N_0 given in (h) a faithful permutation representation of N_0 with stabilizer H_0 has been calculated. It has degree 44,226.

(j) The character table of N_0 has been calculated by means of MAGMA and the faithful permutation representation given in (k).

(i) Similarly, Table 9.6 of the representatives of the 51 conjugacy classes of N_0 has been calculated using Kratzer's algorithm [10]. This completes the proof. □

Lemma 5.3. *Keep the notations of Lemmas* 5.1 *and* 5.2. *Let* $C = \langle d_0, d_1, d_2 \rangle \leqslant H$. *In* $N_0 = N_G(d_0)$ *define the following elements*:

$$y = k_5 d_1 d_2^{-1} k_1 k_2 k_3 k_4 k_6 s_0^{-1} d_1^{-1} s_0^{-1} k_5 d_0^{-1},$$

$$d_9 = d_2^{-1} k_3 d_2 s_0 k_1 d_1 s_0 d_2^{-1} s_0^{-1} d_2 s_0^{-1},$$

$$d_{10} = k_1 k_3^{-1} s_0 k_1 d_2^{-1} s_0 d_1 s_0^{-1} d_1^{-1} s_0^{-1}.$$

Then the following statements hold:

(a) $d_2^y = d_{10}$, $z^y = k_7$, $d_0^y = d_0^2$, *and* $y^2 = 1$.
(b) $K = O[C_G(C)] = C \times D_1$, *where* $D_1 = [K, z] = \langle d_9, d_{10} \rangle$ *is an elementary abelian group of order* 9 *and* $d_{10}^{k_7} = d_{10}$ *and* $d_9^{k_7} = d_9^2$.
(c) $C_G(C) = K : \langle z \rangle$.
(d) $N_G(C) = K N_H(C)$.
(e) $N_G(C)/K \cong N_G(D_1)/C_G(D_1) \cong \mathrm{GL}_2(3)$.

Proof. (a) The element y of order two has been found by means of MAGMA and the faithful permutation representation of $N_0 = N_G(d_0) = (\langle d_0 \rangle \times W) : \langle k_6 \rangle$ given in Lemma 5.2 after having checked that $k_7 \in N_0$.

(b), (c): By Lemma 5.1 $C = \langle d_0, d_1, d_2 \rangle$ is an elementary abelian normal subgroup of the fixed Sylow 3-subgroup $H_3 = \langle d_0, d_1, d_2, d_3 \rangle \cong \mathbb{Z} \wr \mathbb{Z}_3$. Let $C_0 = C_G(d_0)$. Then $C_0 \cong \langle d_0 \rangle \times W$ by Lemma 5.2. Another application of MAGMA and the faithful permutation representation of N_0 mentioned above yields that $C_G(C) = C_{C_0}(C)$ has an elementary abelian normal subgroup K of order $|K| = 3^5$ such that $C_{C_0}(C) = K : \langle z \rangle = (C \times [K, z]) : \langle z \rangle$, and that $D_1 = [K, z] = \langle d_9, d_{10} \rangle$ is elementary abelian of order 9. Furthermore, $C_{D_1}(k_7) = \langle d_{10} \rangle$, and $d_9^{k_7} = d_9^2$.

(d) K is a characteristic subgroup of $C_G(C)$. Thus it is normal in $N_G(C)$. As $C_G(C) = K : \langle z \rangle$ it follows that $N_G(C) = N_H(C)K$.

(e) Hence $N_G(C)/K = N_H(C)K/K \cong N_H(C)/C$. Using MAGMA and the faithful permutation representation of H given in Lemma 5.1 it has been checked that $N_H(C)$ has a semi-dihedral Sylow 2-subgroup, and that $N_H(C)/C \cong \mathrm{GL}_2(3)$.

As $K = C \times D_1$ is normal in $N_G(C)$ it follows that each $x \in N_G(C)$ normalizes D_1. Therefore $N_G(D_1)/C_G(D_1) \cong \mathrm{GL}_2(3)$. This completes the proof. □

Lemma 5.4. *Keep the notations of Lemma* 5.1. *Let* $d_5 = d_1 d_2^{-1}$. *Then the following statements hold*:

(a) $C_4 = C_H(d_4) = \langle d_4 \rangle \times H_4$, *where* $H_4 = \langle d_0, d_5, k_9, k_{10} \rangle$ *has a Sylow* 2*-subgroup* $T_4 = \langle k_9, k_{10} \mid k_9^4 = z = k_{10}^2, k_9^{k_{10}} = k_9^{-1} \rangle$, *and* $N_H(d_4) = \langle C_4, k_7 \rangle = \langle C_4, k_8 \rangle$.

(b) $C_H(d_4) \cap C_H(k_8)$ *has order* $2^3 \cdot 3$.

(c) $R := O[C_G(d_4)] = \langle d_4 \rangle \times B$, *where* $B = \langle b_1, b_2, b_3, b_4 \rangle$ *is an elementary abelian normal subgroup of* $C_G(d_4)$ *with order* $|B| = 3^4$, *and* $B = [R, z]$.

(d) $N_4 = N_G(d_4)$ *is a split extension of its normal subgroup* R *by* $Q_4 := N_H(d_4)/\langle d_4 \rangle = \langle d_0, d_5, k_9, k_{10}, k_8 \rangle$ *described by the following matrices of the generators of* Q_4 *corresponding to their conjugate action on* R *with respect to its basis* $\mathbb{B} = \{d_4, b_i \mid 1 \leqslant i \leqslant 4\}$:

$$(d_0) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 2 & 2 & 0 & 0 \end{pmatrix}, \qquad (d_5) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 & 2 \end{pmatrix},$$

$$(k_9) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 2 & 1 \end{pmatrix}, \qquad (k_{10}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 & 0 \end{pmatrix},$$

$$(k_8) = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(e) $N_4 = N_G(d_4) = \langle d_4, b_1, b_2, b_3, b_4, d_0, d_5, k_9, k_{10}, k_8 \rangle$ *has the following set of defining relations*:

$$\begin{aligned}
&d_4^3 = b_1^3 = b_2^3 = b_3^3 = b_4^3 = d_4^{-1} b_1^{-1} d_4 b_1 = 1, \\
&d_4^{-1} b_2^{-1} d_4 b_2 = b_1^{-1} b_2^{-1} b_1 b_2 = d_4^{-1} b_3^{-1} d_4 b_3 = 1, \\
&b_1^{-1} b_3^{-1} b_1 b_3 = b_2^{-1} b_3^{-1} b_2 b_3 = d_4^{-1} b_4^{-1} d_4 b_4 = 1, \\
&b_1^{-1} b_4^{-1} b_1 b_4 = b_2^{-1} b_4^{-1} b_2 b_4 = b_3^{-1} b_4^{-1} b_3 b_4 = 1, \\
&d_0^{-1} d_4 d_0 d_4^{-1} = d_0^{-1} b_1 d_0 b_2 b_4 = d_0^{-1} b_2 d_0 b_1 b_4 = 1, \\
&d_0^{-1} b_3 d_0 b_2 b_3^{-1} b_4^{-1} = d_0^{-1} b_4 d_0 b_1 b_2 = d_5^{-1} d_4 d_5 d_4^{-1} = 1,
\end{aligned}$$

$$\begin{aligned}
&d_5^{-1}b_1d_5b_1^{-1}b_2b_4^{-1} = d_5^{-1}b_2d_5b_4^{-1} = d_5^{-1}b_3d_5b_1^{-1}b_2^{-1}b_3^{-1}b_4^{-1} = 1,\\
&d_5^{-1}b_4d_5b_2b_4 = k_9^{-1}d_4k_9d_4^{-1} = k_9^{-1}b_1k_9b_2b_3b_4 = 1,\\
&k_9^{-1}b_2k_9b_2^{-1}b_3 = k_9^{-1}b_3k_9b_2^{-1}b_3^{-1} = k_9^{-1}b_4k_9b_1b_2^{-1}b_3b_4^{-1} = 1,\\
&k_{10}^{-1}d_4k_{10}d_4^{-1} = k_{10}^{-1}b_1k_{10}b_1^{-1}b_2^{-1} = k_{10}^{-1}b_2k_{10}b_1^{-1}b_2 = 1,\\
&k_{10}^{-1}b_3k_{10}b_1b_4^{-1} = k_{10}^{-1}b_4k_{10}b_1^{-1}b_2^{-1}b_3 = k_8^{-1}d_4k_8d_4 = 1,\\
&k_8^{-1}b_1k_8b_1^{-1}b_3b_4 = k_8^{-1}b_2k_8b_1^{-1}b_2b_3^{-1}b_4^{-1} = 1,\\
&k_8^{-1}b_3k_8b_3b_4^{-1} = k_8^{-1}b_4k_8b_4^{-1} = d_0^3 = d_5^3 = 1,\\
&k_9^8 = k_{10}^4 = k_8^2 = d_0^{-1}d_5^{-1}d_0d_5 = d_0^{-1}k_9^{-1}d_5k_9 = 1,\\
&d_5^{-1}k_9^{-1}d_0^{-1}k_9 = k_9^{-1}k_{10}^{-1}k_9^{-1}k_{10} = k_{10}^{-2}d_0^{-1}k_{10}^2d_0 = 1,\\
&k_9^{-3}k_{10}^2k_9^{-1} = k_8k_9^{-2}k_8k_{10} = d_0k_8d_5^{-1}d_0^{-1}k_8d_5k_{10}^{-1} = 1,\\
&d_0^{-1}k_{10}^{-1}k_9^{-1}d_0^{-1}k_{10}d_5k_{10} = 1.
\end{aligned}$$

(f) *N_4 has a faithful permutation representation of degree* 243 *with stabilizer* $\{d_0, d_5, k_9, k_{10}, k_7\}$.

(g) $S_4 = \langle d_0, d_1, d_2, b_1, b_2, b_3, b_4\rangle = RC$ *is a Sylow* 3*-subgroup of* $N_4 = N_G(d_4)$ *with center* $Z(S_4) = \langle d_4\rangle \times \langle d_9, d_{10}\rangle$, *where* $C = \langle d_0, d_1, d_2\rangle$, $d_9 = b_1b_2^{-1}$ *and* $d_{10} = b_1b_4^{-1}$.

(h) *A system of representatives n_i and the corresponding centralizer orders $|C_{N_4}(n_i)|$ of the* 48 *conjugacy classes $n_i^{N_4}$ of N_4 is given in Table* 9.7.

(i) *The character table of N_4 is stated in Table* 10.7.

Proof. The statements (a) and (b) hold by Lemma 5.1.

(c) By the same result we know that any Sylow 2-subgroup T_4 of $C_G(d_4)$ is a generalized quaternion group of order 16. Furthermore, $C_H(d_4)/\langle d_4\rangle \cong 2A_6$ and $N_H(d_4)/\langle d_4\rangle \cong 2S_6$. Therefore the Brauer–Suzuki theorem implies that

$$C_G(d_4) = RC_H(d_4), \quad \text{where } R = O\bigl[C_G(d_4)\bigr].$$

By Lemma 5.1 we know that the Klein four group $V = \langle z, k_7\rangle$ operates on R by conjugation and that k_7 and k_7z are conjugate involutions in H. Furthermore, $|C_H(d_4) \cap C_H(k_7)| = 2^3 \cdot 3$. Applying the Brauer–Wielandt theorem we get

$$|R|\bigl|C_R(V)\bigr|^2 = \bigl|C_R(z)\bigr|\bigl|C_R(k_7)\bigr|^2. \qquad (*)$$

By Lemma 3.1 G has a unique conjugacy class of involutions. Therefore $|C_X(k_7)| \leqslant |C_G(z)| = |H|$. Since $C_H(d_4)/\langle d_4\rangle = 2A_6$ it follows that $C_X(z) = \langle d_4\rangle$. Clearly, $C_R(V) \leqslant C_R(z) = \langle d_4\rangle$. As d_4 is inverted by k_7 it follows that $C_R(V) = 1$.

Let $\bar{R} = R/\langle d_4\rangle$. Then $\bar{R} = R/C_R(z)$ is inverted by z. Hence $\bar{R}$ is an abelian group by Burnside's lemma. Thus $\bar{R} = \bar{R}_3 \times \bar{R}_5 \times \bar{R}_7$, where $\bar{R}_p$ denotes the Sylow p-subgroup of $\bar{R}$ for each prime $p \in \{3, 5, 7\}$. Since $N_H(d_4)/\langle d_4\rangle \cong 2S_6$ it follows that each p-socle $\Omega(\bar{R}_p)$

of $\bar{R}$ is a faithful F_p-module for $p \in \{3, 5, 7\}$, where $F_p = \mathbb{Z}/p\mathbb{Z}$. However, the smallest dimension of a faithful irreducible $F_p[2S_6]$-module is 4, and if $p = 3$, then the second smallest dimension is 12. Now $(*)$ implies that $|\bar{R}_p| \leqslant |C_R(k_7)|^2 = p^2$ for each prime $p \in \{5, 7\}$. Hence $\bar{R}$ is an abelian 3-group. Since a Sylow 3-subgroup of H is isomorphic to the ones of the alternating group A_9, it is a wreathed product of order 81. Hence $|C_R(k_7)|_3 \leqslant 3^3$, and $|\bar{R}| \leqslant 3^6$. In particular, $\Omega(\bar{R}_3)$ is elementary abelian of order 3^4. As $|\bar{R} : \Omega(\bar{R}_3)| \leqslant 3^2$, and z inverts $\bar{R}/\Omega(\bar{R}_3)$ it follows that $|\bar{R}| = 3^4$.

Hence $R/\langle d_4\rangle \cong B = \langle b_1, b_2, b_3, b_4\rangle$ as $F_3[2S_6]$-modules. By the 3-modular character table of $2S_6$ we know that B does not belong to the principal 3-block. Hence R splits over $\langle d_4\rangle$, and so we may assume that $R = \langle d_4\rangle \times B$, because d_4 is in the center of $C_G(d_4)$.

(d) $L = N_H(d_4)/\langle d_4\rangle \cong 2S_6$ has a unique 4-dimensional faithful irreducible module over $F_3 = \mathbb{Z}/3\mathbb{Z}$. By (c) the normal subgroup $B = \langle b_1, b_2, b_3, b_4\rangle$ is such a faithful irreducible F_3L-module, on which L acts by conjugation. With respect to its basis $\mathbb{B} = \{b_1, b_2, b_3, b_4\}$ the generators of $L = \langle d_0, d_5, k_9, k_{10}, k_8\rangle$ have the matrices as given in statement (d).

Clearly, in $C_H(d_4) = \langle d_4\rangle \times \langle d_0, d_5, k_9, k_{10}\rangle$ the abelian 3-group $C = \langle d_0, d_1, d_2\rangle = \langle d_4\rangle \times \langle d_0, d_5\rangle$ is a Sylow 3-subgroup which splits over $\langle d_4\rangle$. Hence $S_4 = R\langle d_0, d_5\rangle$ is a Sylow 3-subgroup of $C_G(d_4)$ by (c) which splits over $R = \langle d_4\rangle \times B$. Therefore $N_4 = N_G(d_4)$ is a split extension of R by L by Gaschütz' theorem. Using now MAGMA it is easy to construct the unique split extension $N_4 = R : L$ by means of the matrices of the generators of L.

(e) The output of MAGMA is the presentation of N_4 given in statement (e).

(f) Clearly, $L = \langle d_0, d_5, k_9, k_{10}, k_7\rangle$ can be chosen as a stabilizer of a faithful permutation representation of N_4. It has degree 3^5.

(g) Let $C = \langle d_0, d_1, d_2\rangle$. Then $K = O[C_G(C)] = C \times D_1$ by Lemma 5.3, where $D_1 = \langle d_9, d_{10}\rangle$. Clearly, $C_G(C) \leqslant C_G(d_4)$. Furthermore, $S_4 = RC$ is a Sylow 3-subgroup of N_4, because R is normal in N_4. By (d) $R = \langle d_4\rangle \times B$, and $B - \{1\}$ consists of all elements of R which are inverted by z. Since B is a normal subgroup of N_4 it is normalized by k_7. Using MAGMA and the faithful permutation representation of N_4 given in (f) it has been checked that $\langle b_1 b_4^{-1}\rangle$ is the only cyclic subgroup of B which is centralized by k_7, and that $\langle b_2 b_4^{-1}\rangle$ is the only cyclic subgroup of B which is inverted by k_7. Let $c_5 = b_1 b_4^{-1}$ and $c_6 = b_2 b_4^{-1}$.

By Lemma 5.3 we know that $K = O[C_G(C)] = C \times D_1$, where $D_1 = \langle d_9, d_{10}\rangle$ and $d_9^{k_7} = d_9^{-1}$, $d_{10}^{k_7} = d_{10}$. As $K = O[C_G(C)] \leqslant O[C_{N_4}(C)]$ we calculate the latter group by means of MAGMA and the faithful permutation representation of N_4 given in (f). It yields that $O[C_{N_4}(C)] = C \times \langle c_5\rangle \times \langle c_6\rangle$. Hence $K = C \times D_1 = C \times \langle c_5, c_6\rangle$. Since both c_5 and d_{10} are centralized by k_7 and both elements c_6 and d_9 are inverted by k_7, we may assume that $c_5 = b_1 b_4^{-1} = d_{10}$ and $c_6 = b_1 b_2^{-1} = d_9$. In particular, $D_1 = \langle c_5, c_6\rangle = \langle d_{10}, d_9\rangle \leqslant B \leqslant R = \langle d_4\rangle \times B$.

Another application of MAGMA and the faithful permutation representation of N_4 given in (f) yields that the center $Z(S_4)$ of the Sylow 3-subgroup S_4 has order 27. Since $S_4 = RC$, and D_1 commutes with all elements of R and C it follows that $Z(S_4) = \langle d_4\rangle \times D_1$.

The final assertions (h) and (i) have been shown by means of MAGMA and the faithful permutation representation of N_4 given in (f). This completes the proof. □

Lemma 5.5. *Keep the notations of the previous lemmas. Let*:

$$\begin{aligned}
&d_{14} = d_1k_1k_2k_3s_0d_2^{-1}s_0^{-1}d_1s_0^{-1}d_1^{-1} \in N_0,\\
&t = k_9^2k_7, \qquad d_{15}d_{14}^t, \qquad b_5 = b_2b_3 \quad \text{and} \quad b_6b_2b_3^{-1}.
\end{aligned}$$

Let $M = C_G(D_1)$. Then the following assertions hold:

(a) *t has order 2, and 3 is the order of d_{14}, d_{15}, b_5 and b_6.*
(b) *$d_0^t = d_0^2$, $d_1^t = d_2^2$, $d_9^t = d_{10}$, $b_6^t = b_5^2$, $k_7^t = k_7z$.*
(c) *$Y_0 = C_{N_0}(D_1) = \langle d_0, d_1, d_2, d_9, d_{10}, d_{14}, d_{15}\rangle$ has the following set $\mathcal{R}(Y_0)$ of defining relations*:

$$\begin{aligned}
&d_0^3 = d_1^3 = d_2^3 = d_9^3 = d_{10}^3 = d_{14}^3 = d_{15}^3 = 1,\\
&[d_0, d_1] = [d_0, d_2] = [d_0, d_9] = [d_0, d_{10}] = [d_0, d_{14}] = [d_0, d_{15}] = 1,\\
&[d_1, d_2] = [d_1, d_9] = [d_1, d_{10}] = [d_1, d_{15}] = 1,\\
&[d_2, d_9] = [d_2, d_{10}] = [d_2, d_{14}] = 1,\\
&[d_9, d_{10}] = [d_9, d_{14}] = [d_9, d_{15}] = 1,\\
&[d_{10}, d_{14}] = [d_{10}, d_{15}] = 1,\\
&d_{14}^{-1}d_{10}^{-1}d_1^{-1}d_{14}d_1 = d_{15}d_9^{-1}d_2^{-1}d_{15}^{-1}d_2 = 1,\\
&d_{15}^{-1}d_{10}d_9^{-1}d_2^{-1}d_1^{-1}d_{14}d_{15}d_{14}^{-1} = 1.
\end{aligned}$$

(d) *$Y_4 = C_{N_4}(D_1) = \langle d_0, d_1, d_2, d_9, d_{10}, b_5, b_6\rangle$ has the following set $\mathcal{R}(Y_4)$ of defining relations*:

$$\begin{aligned}
&d_0^3 = d_1^3 = d_2^3 = d_9^3 = d_{10}^3 = b_5^3 = b_6^3 = 1,\\
&[d_0, d_1] = [d_0, d_2] = [d_0, d_9] = [d_0, d_{10}] = 1,\\
&[d_1, d_2] = [d_1, d_9] = [d_1, d_{10}] = 1,\\
&[d_2, d_9] = [d_2, d_{10}] = 1,\\
&[d_9, d_{10}] = [d_9, b_5] = [d_9, b_6] = 1,\\
&[d_{10}, b_5] = [d_{10}, b_6] = [b_5, b_6] = 1,\\
&b_6d_9^{-1}d_0^{-1}b_6^{-1}d_0 = b_5d_{10}^{-1}d_0^{-1}b_5^{-1}d_0 = b_5^{-1}d_9^{-1}d_1^{-1}b_5d_1 = 1,\\
&b_6d_{10}^{-1}d_1^{-1}b_6^{-1}d_1 = b_5d_9^{-1}d_2^{-1}b_5^{-1}d_2 = b_6^{-1}d_{10}^{-1}d_2^{-1}b_6d_2 = 1.
\end{aligned}$$

(e) *$M = \langle C_M(z), C_M(k_7), C_M(k_7z)\rangle$, $C_M(z) = \langle d_0, d_1, d_2\rangle = C$, $C_M(k_7z) = \langle d_0, d_{10}, d_{15}, b_5\rangle$, and $C_M(k_7) = [C_M(k_7z)]^t = \langle d_0, d_9, d_{14}, b_6\rangle$.*

Table 5.1

x	d_0	d_1	d_2	d_9	d_{10}	d_{14}	d_{15}	b_5	b_6
x^{k_7}	d_0	d_1^{-1}	d_2^{-1}	d_9^{-1}	d_{10}	d_{14}^{-1}	d_{15}	b_5	b_6^{-1}
x^z	d_0	d_1	d_2	d_9^{-1}	d_{10}^{-1}	d_{14}	d_{15}^{-1}	b_5^{-1}	b_6^{-1}
x^t	d_0^{-1}	d_2^{-1}	d_1^{-1}	d_{10}	d_9	d_{15}	d_{14}	b_6^{-1}	b_5^{-1}

Table 5.2

x	d_0	d_1	d_2	d_9	d_{10}	d_{14}	d_{15}	b_5	b_6
x^{k_7}	d_0	d_1^{-1}	d_2^{-1}	d_9^{-1}	d_{10}	d_{14}^{-1}	d_{15}	b_5	b_6^{-1}
x^z	d_0	d_1	d_2	d_9^{-1}	d_{10}^{-1}	d_{14}	d_{15}^{-1}	b_5^{-1}	b_6^{-1}

(f) *Besides the relations in* (c) *and* (d) *the generators of* $C_M(k_7) = \langle d_0, d_9, d_{14}, b_6\rangle$ *and* $C_M(k_7z) = \langle d_0, d_{10}, d_{15}, b_5\rangle$ *satisfy the additional relations*

$$b_6 d_{14}^{-1} d_0^{-1} b_6^{-1} d_{14} = 1, \qquad b_5^{-1} d_{15} d_0^{-1} b_5 d_{14}^{-1} = 1.$$

(g) $|M| = 3^9$ *and* $Z(M) = D_1$.

(h) M/K *is elementary abelian of order* 3^4, *where* $K = C \times D_1$.

(i) *The involutions* z, k_7 *and* t *operate on*

$$M = \langle d_0, d_1, d_2, d_9, d_{10}, d_{14}, d_{15}, b_5, b_6\rangle$$

by conjugation as shown in Table 5.1.

(j) *The group* $M = C_G(D_1) = \langle d_0, d_1, d_2, d_9, d_{10}, d_{14}, d_{15}, b_5, b_6\rangle$ *has a set* $\mathcal{R}(M)$ *of defining relations consisting of* $\mathcal{R}(Y_0)$, $\mathcal{R}(Y_4)$ *and the following relations*:

$$[b_6, d_{14}] = d_0^2, \qquad \left[b_5^2, d_{15}\right] = d_0,$$
$$[b_6, d_{15}] = d_2^2 d_9 d_{10}^2, \qquad \left[b_5^2, d_{14}\right] = d_1 d_{10} d_9^2,$$
$$\left[b_6^2, d_{15}\right] = d_2 d_9^2 d_{10}^2, \qquad \left[b_6^2, d_{15}^2\right] = d_2^2 d_9^2 d_{10}.$$

(k) $U = \langle z, k_7, t\rangle$ *is a dihedral group of order* 8, *and the subgroup* $MU \leqslant N_G(D_1)$ *is uniquely determined by* H *up to isomorphism.*

Proof. (a) Using the faithful permutation representations of N_0 and N_4 given in Lemmas 5.2 and 5.4 and MAGMA one can easily verify that $d_{14} \in C_{N_0}(D_1)$, $d_{14}^3 = 1$, $t \in N_{N_4}(D_1)$, $t^2 = 1$, and that $b_5, b_6 \in C_{N_4}(D_1)$, $b_6^t = b_5^{-1}$, $b_6^3 = b_5^3 = 1$. Hence (a) holds.

(b) can be checked similarly.

(c) and (d) are proved computationally by the methods described in (a).

(e) The involutions z and k_7 of G operate on the generators of Y_0 and Y_4 as described in Table 5.2.

By Lemma 5.3 d_{10} and d_2 are conjugate in G. Hence $C_G(d_{10}) \cong C_G(d_2)$. From Lemma 5.1 and Brauer–Suzuki's theorem follows that

$$L_2 := C_G(d_2) = O(L_2)C_H(d_2) \quad \text{and} \quad C_H(d_2) \cong C : \mathsf{SL}_2(3),$$

where $C = \langle d_0, d_1, d_2\rangle$ is elementary abelian of order 27. Let $X = O(L_2)$. Then $V = \langle z, k_7\rangle$ operates on X by conjugation and k_7 and k_7z are conjugate in $N_H(d_2)$ by Lemma 5.1. Furthermore,

$$C_X(V) = X \cap C_G(z) \cap C_G(k_7) = C \cap C_H(k_7) = \langle d_0\rangle$$

by Table 5.2. Theorem of Brauer and Wielandt states that

$$|X|\big|C_X(V)\big|^2 = \big|C_X(z)\big|\big|C_X(k_7)\big|\big|C_X(k_7z)\big| = \big|C_X(z)\big|\big|C_X(k_7)\big|^2. \tag{$*$}$$

As $|C_H(d_2)| = 2^3 \cdot 3^4$ and $C_H(d_2)/O[C_H(d_2)] = \mathsf{SL}_2(3)$ we know that $C_X(z) = C$ has order 27. The involution z operates fixed point freely on $\bar{X} := X/C_X(z)$. Therefore $\bar{X}$ is a faithful $\mathsf{SL}_2(3)$-module. Since all involutions of G are conjugate, and all Sylow p-subgroups of H for $p \in \{5, 7\}$ are cyclic, it follows from the character table of $\mathsf{SL}_2(3)$ and Maschke's theorem that $\bar{X}$ is a 3-group. Therefore $C_X(k_7)$ is a 3-group of order $|C_X(k_7)| \leqslant 3^4$ by Lemmas 2.2 and 3.1. Hence $|X| \leqslant 3^9$ by $(*)$. Furthermore, $C_G(d_2)/X = \mathsf{SL}_2(3)$ implies that $|C_G(d_2)| \leqslant 2^3 \cdot 3^{10}$.

Lemma 5.3 states that $N_G(D_1)/C_G(D_1) \cong \mathrm{GL}_2(3)$, and that $d_2^y = d_{10}$ for some involution $y \in N_0 = N_G(d_0)$. Thus $|C_G(d_2)| = |C_G(d_{10})|$. As $M = C_G(D_1) \leqslant C_G(d_{10})$ it follows that $|M| \leqslant 3^9$, and $C_M(z) = C = \langle d_0, d_1, d_2\rangle$.

Clearly, $M \geqslant \langle C_M(z), C_M(k_7), C_M(k_7z)\rangle$. Since the involutions z and k_7 operate on Y_0 and Y_4 it follows from the relations given in (c) and (d) and Table 5.2 that $C_M(k_7z) \geqslant \langle d_0, d_9, d_{14}, b_6\rangle = T$. Furthermore, by (c) and (d) the subgroup T satisfies the following set $S(T)$ of relations:

$$\begin{aligned} &d_i^3 = 1 = b_6^3 \quad \text{for } i = 0, 9, 14,\\ &[d_0, d_9] = [d_0, d_{14}] = [d_9, d_{14}] = [d_9, b_6] = 1, \quad \text{and}\\ &b_6 d_9^{-1} d_0^{-1} b_6^{-1} d_0 = 1. \end{aligned}$$

Hence $\langle d_0, d_9, d_{14}\rangle$ is elementary abelian of order 27, and does not contain b_6. Since all involutions of G are conjugate by Lemma 3.1 it follows now from Lemma 2.2 that

$$T = \langle d_0, d_9, d_{14}, b_6\rangle = C_M(k_7z) \cong \mathbb{Z}_3 \wr \mathbb{Z}_3.$$

Let $d_{15} = d_{14}^t$. Then (b) and these equations imply that

$$C_M(k_7) = \big[C_M(k_7)\big]^t = \big\langle d_0^t, d_9^t, d_{14}^t, b_6^t\big\rangle = \big\langle d_0^1, d_{10}, d_{15}, b_5^{-1}\big\rangle = \langle d_0, d_{10}, d_{15}, b_5\rangle.$$

Therefore $|C_M(z)| = 27$, $|C_M(k_7)| = |C_M(k_7z)| = 81$, and

$$C_M(z) \cap C_M(k_7) = \langle d_0 \rangle = C_M(z) \cap C_M(k_7z) = C_M(k_7) \cap C_M(k_7z).$$

As $|M| \leqslant 3^9$ it follows that $M = \langle C_M(z), C_M(k_7), C_M(k_7z) \rangle$, and $|M| = 3^9$. Hence (e) and the first equation of (g) hold.

(f) Applying MAGMA to the finitely generated group $T = \langle d_0, d_9, d_{14}, b_6 \rangle$ of order 81 it follows that T has a set $\mathcal{R}(T)$ of defining relations consisting of $\mathcal{S}(T)$ and the following relation: $b_6 d_{14}^{-1} d_0^{-1} b_6^{-1} d_{14} = 1$. The second statement of (f) holds now by (b) and (e).

(g) By the proof of (e) it remains to show that $Z(M) = D_1$. Clearly, $D_1 = \langle d_9, d_{10} \rangle \leqslant Z(M)$. As $d_9 = b_1 b_2^{-1}$, $d_{10} = b_1 b_4^{-1}$, $b_5 = b_2 b_3$ and $b_6 = b_2 b_3^{-1}$ it follows that the elementary abelian group $R = \langle b_1, b_2, b_3, b_4 \rangle = \langle d_9, d_{10}, b_5, b_6 \rangle$. Hence $S_4 = CR = \langle d_0, d_1, d_2, d_9, d_{10}, b_5, b_6 \rangle$ is a Sylow 3-subgroup of $N_4 = N_G(d_4)$ with center $Z(S_4) = \langle d_4 \rangle \times D_1$ by Lemma 5.4, where $d_4 = d_1 d_2$. By (c) the following equations hold: $d_{14}^{d_1} = d_{10} d_{14}$, $[d_2, d_{10}] = [d_2, d_{14}] = 1$. Hence $d_4^{-1} d_{14} d_4 = d_{10} d_{14} \neq d_{14}$, because $d_{10} \neq 1$. Therefore, $d_4 \notin Z(M)$, and assertion (g) holds.

(h) $K = C \times D_1 = \langle d_0, d_1, d_2, d_9, d_{10} \rangle$ is a normal subgroup of M, because $Z(M) = \langle d_9, d_{10} \rangle$ and (b) and the relations given in (c) and (d) imply that:

$$\begin{aligned} &d_0^{d_{14}} = d_0^{d_{15}} = d_0, \qquad && d_0^{b_5} = d_0 d_{10} \quad \text{and} \quad && d_0^{b_6} = d_0 d_9, \\ d_1^{d_{14}} = d_1 d_{10}, \qquad &d_1^{d_{15}} = d_1, && d_1^{b_5} = d_1 d_9^2 \quad \text{and} \quad && d_1^{b_6} = d_1 d_{10}, \\ d_2^{d_{14}} = d_2, \qquad &d_2^{d_{15}} = d_2 d_9^2, && d_2^{b_5} = d_2 d_9^2 \quad \text{and} \quad && d_2^{b_6} = d_2 d_{10}^2. \end{aligned}$$

From (b) and (e) follows that the involution z operates on M by conjugation. As $C_M(z) = C < K$ the involution z inverts the factor group M/K. Hence M/K is abelian by Burnside's lemma. Thus M/K is an elementary abelian group of order 3^4 by (c), (d), and (g).

(i) This assertion follows immediately from (h), (b) and the table given in the proof of statement (e).

(j) By (h) the group M/K is elementary abelian and generated by the residue classes of d_{14}, d_{15}, b_5 and b_6. Therefore for each pair $x, y \in \{d_{14}, d_{15}, b_5, b_6\}$ the commutator $[x, y] \in K = C \times D_1 = \langle d_0, d_1, d_2, d_9, d_{10} \rangle$. In particular, there is a 5-tuple of integers $c_i \in \{0, 1, 2\}$, $1 \leqslant i \leqslant 5$, such that

(1) $[b_6, d_{15}] = d_0^{c_1} d_1^{c_2} d_2^{c_3} d_9^{c_4} d_{10}^{c_5}$.

Now (i) implies that $[b_6, d_{15}]^t = [b_5^2, d_{14}]$ and so:

(2) $[b_5^2, d_{14}] = (d_0^{c_1} d_2^{c_2} d_1^{c_3})^2 d_{10}^{c_4} d_9^{c_5}$.

Using the equations $[b_6, d_{15}]^{k_7} = [b_6^2, d_{15}]$ and $[b_6, d_{15}]^z = [b_6^2, d_{15}^2]$ the following equations hold by (i):

(3) $[b_6^2, d_{15}] = d_0^{c_1} (d_1^{c_1} d_2^{c_3} d_9^{c_4})^2 d_{10}^{c_5}$,
(4) $[b_6^2, d_{15}^2] = d_0^{c_1} d_1^{c_2} d_2^{c_3} (d_9^{c_4} d_{10}^{c_5})^2$.

Since b_6 and d_{14} are centralized by k_7 it follows that $[b_6, d_{14}] \in C_M(k_7) \cap K = \langle d_0, d_9\rangle$. Hence there are integers $a_1, a_2 \in \{0, 1, 2\}$ such that

(5) $[b_6, d_{14}] = d_0^{a_1} d_9^{a_2}$.

Since $[b_6, d_{14}]^t = [b_5^2, d_{15}]$ by (i) it follows that

(6) $[b_5^2, b_{15}] = d_0^{2a_1} d_{10}^{a_2}$.

Therefore for each 7-tuple $\tau = (a_1, a_2, c_1, c_2, c_3, c_4, c_5) \in [0, 1, 2]^7$ there is a finitely presented group

$$M_\tau = \langle d_0, d_1, d_2, d_9, d_{10}, d_{14}, d_{15}, b_5, b_6\rangle$$

satisfying the relations $\mathcal{R}(Y_0)$, $\mathcal{R}(Y_4)$ and the corresponding relations (1)–(6). Hence there are 3^7 such groups M_τ. Using MAGMA it has been checked that $|M_T| \leqslant 3^9$ for each τ, and that $|M_\tau| = 3^9$ for exactly the following nine 7-tuples:

$$\begin{array}{lll}
(0,0,0,0,0,0,0), & (0,1,0,0,0,0,0), & (0,2,0,0,0,0,0),\\
(1,0,0,0,1,2,1), & (1,1,0,0,1,2,1), & (1,2,0,0,1,2,1),\\
(2,0,0,0,2,1,2), & (2,1,0,0,2,1,2), & (2,2,0,0,2,1,2).
\end{array}$$

By Lemma 5.3 we know that $N_G(D_1)/C_G(D_1) \cong \mathrm{GL}_2(3)$. In particular, each possible group $M_\tau \cong C_G(D_1)$ must have an automorphism group $\mathrm{Aut}(M_\tau)$ which contains a semi-dihedral 2-subgroup of order 16.

Using MAGMA it has been checked that in the first 6 cases a Sylow 2-subgroup of $\mathrm{Aut}(M_\tau)$ has order 8, and that $\mathrm{Aut}(M_\tau)$ has a semi-dihedral Sylow 2-subgroup of order 16 in the last 3 cases. Furthermore, these 3 groups M_τ of order 3^9 are isomorphic by application of the isomorphism test for p-groups implemented in MAGMA. Hence we may take the relations of the 7-tuple $(2, 0, 0, 0, 2, 1, 2)$, which confirms statement (j).

(k) By (b) the 2-group $U = \langle z, k_7, t\rangle$ is dihedral of order 8. Its action on $M = C_G(D_1)$ is uniquely determined by (i). Hence the semi-direct product $MU \leqslant N_G(D_1)$ is uniquely determined by (j). This completes the proof. □

Lemma 5.6. *Keep the notations of the previous lemmas. Then the following statements hold*:

(a) $N_G(D_1)$ *is a uniquely determined extension of* $M = C_G(D_1)$ *by* $\langle k_9, d_3\rangle \cong \mathrm{GL}_2(3)$.

(b) $N_1 = N_G(D_1) = \langle M, k_9, d_3\rangle = \langle d_0, d_1, d_2, d_9, d_{10}, d_{14}, d_{15}, b_5, b_6, k_9, d_3\rangle$ *has he following set* $\mathcal{R}(N_1)$ *of defining relations*:

$$d_0^3 = d_1^3 = d_2^3 = d_9^3 = d_{10}^3 = d_{14}^3 = d_{15}^3 = d_0^{-1} d_1^{-1} d_0 d_1 = 1,$$
$$d_0^{-1} d_2^{-1} d_0 d_2 = d_1^{-1} d_2^{-1} d_1 d_2 = d_0^{-1} d_9^{-1} d_0 d_9 = 1,$$

$$
\begin{aligned}
&d_1^{-1}d_9^{-1}d_1d_9 = d_2^{-1}d_9^{-1}d_2d_9 = d_0^{-1}d_{10}^{-1}d_0d_{10} = 1,\\
&d_1^{-1}d_{10}^{-1}d_1d_{10} = d_2^{-1}d_{10}^{-1}d_2d_{10} = d_9^{-1}d_{10}^{-1}d_9d_{10} = 1,\\
&d_0^{-1}d_{14}^{-1}d_0d_{14} = d_2^{-1}d_{14}^{-1}d_2d_{14} = d_9^{-1}d_{14}^{-1}d_9d_{14} = 1,\\
&d_{10}^{-1}d_{14}^{-1}d_{10}d_{14} = d_0^{-1}d_{15}^{-1}d_0d_{15} = d_1^{-1}d_{15}^{-1}d_1d_{15} = 1,\\
&d_9^{-1}d_{15}^{-1}d_9d_{15} = d_{10}^{-1}d_{15}^{-1}d_{10}d_{15} = d_{14}^{-1}d_{10}^{-1}d_1^{-1}d_{14}d_1 = 1,\\
&d_{15}d_9^{-1}d_2^{-1}d_{15}^{-1}d_2 = d_{15}^{-1}d_{10}d_9^{-1}d_2^{-1}d_1^{-1}d_{14}d_{15}d_{14}^{-1} = 1,\\
&b_5^3 = b_6^3 = d_9^{-1}b_5^{-1}d_9b_5 = d_{10}^{-1}b_5^{-1}d_{10}b_5 = 1,\\
&d_9^{-1}b_6^{-1}d_9b_6 = d_{10}^{-1}b_6^{-1}d_{10}b_6 = b_5^{-1}b_6^{-1}b_5b_6 = 1,\\
&b_6d_9^{-1}d_0^{-1}b_6^{-1}d_0 = b_5d_{10}^{-1}d_0^{-1}b_5^{-1}d_0 = b_5^{-1}d_9^{-1}d_1^{-1}b_5d_1 = 1,\\
&b_6d_{10}^{-1}d_1^{-1}b_6^{-1}d_1 = b_5d_9^{-1}d_2^{-1}b_5^{-1}d_2 = b_6^{-1}d_{10}^{-1}d_2^{-1}b_6d_2 = 1,\\
&d_{14}^{-1}b_6^{-1}d_{14}b_6d_0^2 = d_{15}^{-1}b_5^{-2}d_{15}b_5^2d_0^4 = 1,\\
&d_{15}^{-1}b_6^{-1}d_{15}b_6d_2^2d_9d_{10}^2 = d_{14}^{-1}b_5^{-2}d_{14}b_5^2d_1^4d_{10}d_9^2 = 1,\\
&d_{15}^{-1}b_6^{-2}d_{15}b_6^2d_2^2d_9d_2^2d_9d_{10}^2 = d_{15}^{-2}b_6^{-2}d_{15}^2b_6^2d_2^2d_9d_{10}^2d_9d_{10}^2 = 1,\\
&k_9^{-1}d_0^{-1}k_9b_6^{-1}d_2d_9b_6d_1^{-1} = k_9^{-1}d_1^{-1}k_9d_9d_0^{-1}d_1^{-1}d_2^{-1} = 1,\\
&k_9^{-1}d_2^{-1}k_9d_0d_1^{-1}d_2^{-1}d_9^{-1} = k_9^{-1}d_9^{-1}k_9d_9d_{10}^{-1} = 1,\\
&k_9^{-1}d_{10}^{-1}k_9d_9d_{10} = k_9^{-1}b_5^{-1}k_9b_6b_5^{-1}d_0d_1^{-1}d_2^{-1}d_{10}^{-1} = 1,\\
&k_9^{-1}b_6^{-1}k_9b_5^{-1}b_6^{-1}d_2d_{10} = k_9^{-1}d_{14}^{-1}k_9b_6^{-1}d_1d_{15}^{-1}b_5^{-1}d_{14} = 1,\\
&k_9^{-1}d_{15}^{-1}k_9b_5b_6d_0d_{14}d_{15}b_6 = d_3^{-1}d_0^{-1}d_3b_5d_0^{-1}d_2^{-1}b_5^{-1} = 1,\\
&d_3^{-1}d_1^{-1}d_3d_2 = d_3^{-1}d_2^{-1}d_3d_1d_2d_9d_0^{-1} = d_3^{-1}d_9^{-1}d_3d_{10} = 1,\\
&d_3^{-1}d_{10}^{-1}d_3d_9^{-1}d_{10}^{-1} = d_3^{-1}b_5^{-1}d_3b_6b_5^{-1}d_0d_9d_{15} = 1,\\
&d_3^{-1}b_6^{-1}d_3b_6b_5^{-1}d_0^{-1}b_5d_{14}^{-1}d_{15}^{-1} = d_3^{-1}d_{14}^{-1}d_3b_5^{-1}b_6^{-1}d_1d_{14}d_0^{-1}d_{15}^{-1} = 1,\\
&d_3^{-1}d_{15}^{-1}d_3b_5^{-1}d_1d_2d_{14}^{-1}b_5 = d_3^3 = k_9d_3^{-1}k_9d_3^{-1} = 1,\\
&k_9^8 = k_9d_3k_9^3d_3k_9^2 = 1.
\end{aligned}
$$

(c) *$N_1 = N_G(D_1)$ has a faithful permutation representation of degree 3^9 with stabilizer $\langle k_9, d_3\rangle \cong \mathrm{GL}_2(3)$.*

(d) *$S_1 = \langle M, d_3\rangle$ is a Sylow 3-subgroup of the simple group G of* Th*-type, and $|S_1| = 3^{10}$.*

(e) *The Sylow 3-subgroup S_1 of G has center $Z(S_1) = \langle d_{10}\rangle$, normalizer $N_G(S_1) = S_1 : \langle z, k_7\rangle$, and D_1 is the unique elementary abelian normal subgroup of order 9 in S_1.*

(f) *A system of representatives n_i of the 52 conjugacy classes $n_i^{N_1}$ of $N_1 = N_G(D_1)$ and the corresponding centralizer orders $|C_{N_1}(n_i)|$ is given in Table 9.8.*

(g) *The character table of $N_1 = N_G(D_1)$ is stated in Table 10.8.*

Proof. (a) By Lemma 5.5 the center of M equals $D_1 = \langle d_9, d_{10}\rangle$. Hence $N_G(M) \leqslant N_G(D_1)$. Using the faithful permutation representation of H given in Lemma 2.2 one can check that $N_H(C) = C : \langle k_9, d_3\rangle$, where $C = \langle d_0, d_1, d_2\rangle$. Therefore Lemma 5.3 implies that

$$N_G(D_1)/M = N_G(D_1)/C_G(D_1) \cong N_H(C)/C \cong \langle k_9, d_3\rangle \cong \mathrm{GL}_2(3).$$

By Lemma 5.5 we may assume that the dihedral group $U = \langle z, k_7, t\rangle$ of order 8 is a subgroup of $N_G(M)/M$, and hence of $N_H(C)/C$. Using MAGMA and the faithful permutation representation of H again it follows that

$$\langle U, d_3\rangle = \langle z, k_7, t, d_3\rangle = \langle k_9, d_3\rangle = \mathrm{GL}_2(3).$$

Therefore $N_G(M) = N_G(D_1)$, and $S_1 = \langle M, d_3\rangle$ is a Sylow 3-subgroup of G.

Another application of MAGMA yields that $\mathrm{Aut}(M)$ has a semi-dihedral Sylow 2-subgroup of order 16 which belongs to a subgroup Λ of $\mathrm{Aut}(M)$ such that the image of M in $\mathrm{Aut}(M)$ has trivial intersection with Λ, and $\Lambda \cong \mathrm{GL}_2(3)$. Therefore we may assume that $N_1 = N_G(D_1) = N_G(M) = \langle M, k_9, d_3\rangle$, and that k_9 and d_3 operate on M by conjugation as their images $\mathrm{im}(k_9)$ and $\mathrm{im}(d_3)$ in $\mathrm{Aut}(M)$ operate on M.

Thus $N_G(D_1)/D_1 = N_G(M)/Z(M)$ is uniquely determined up to isomorphism. Since $D_1 = Z(M)$ is elementary abelian of order 9 we can now apply the algorithm of Cannon and Holt implemented in MAGMA, and find that the group $N_G(D_1) = \langle M, k_9, d_3\rangle$ is uniquely determined up to isomorphism.

(b) Furthermore, this application of MAGMA also provides the presentation of $N_1 = N_G(D_1)$ given in statement (b).

(c) From the faithful permutation representation of N_1 follows by means of MAGMA that N_1 is a split extension of M by $\langle k_9, d_3\rangle \cong \mathrm{GL}_2(3)$. As $|M| = 3^9$ statement (c) holds.

(d) This assertion has already been shown in the proof of (a).

(e) As $Z(M) = \langle d_9, d_{10}\rangle$ and d_9 does not commute with d_3 by the relations given in (b) it follows that $Z(S_1) = \langle d_{10}\rangle$. By the proof of statement (e) of Lemma 5.5 we know that $N_G(d_2)$ and $N_G(d_{10})$ are conjugate in G and that $N_G(d_2)/X \cong \mathrm{GL}_2(3)$, where $X = O[C_G(d_2)]$ has order at most 3^9. Hence $|X| = 3^9$ by (a), and $N_G(S_1)$ is isomorphic to a subgroup of $N_G(d_{10})$. Thus $N_G(S_1) = S_1 : \langle z, k_7\rangle$ by Lemma 5.5. Furthermore, D_1 is the only noncyclic elementary abelian normal subgroup of order 9 in S_1 as has been checked by means of MAGMA and the faithful permutation representation of N_1 described in (c).

(f) The representatives n_i of the 52 conjugacy classes of $N_1 = N_G(D_1)$ have been determined by means of algorithm [10], MAGMA and the faithful permutation representation of N_1 given in (c).

(g) The character table of N_1 has been obtained by application of MAGMA and the faithful permutation representation of N_1.

This completes the proof. □

Lemma 5.7. *Let* $S_1 = \langle M, d_3\rangle$ *be the Sylow* 3*-subgroup of* G *with center* $Z(S) = \langle d_{10}\rangle$ *constructed in Lemma* 5.6. *Then the following statements hold*:

(a) *The Frattini subgroup* $\Phi(S_1)$ *of* S_1 *has order* 3^7.
(b) S_1 *has a uniquely determined maximal subgroup* L *of order* 3^9*suchthat* $L = O[N_G(d_{10})]$.
(c) $N_H(d_2) = C : J_1$, *where*$J_1 = \langle k_1, k_2, d_3, k_7\rangle \cong \mathrm{GL}_2(3)$, *and* $C = \langle d_0, d_1, d_2\rangle = O[N_H(d_2)]$.
(d) $N_{10} = N_G(d_{10}) = LJ$ *and* $L \cap J = 1$, *where* $J = J_1^y$ *and* y *is the involution of* $N_0 = N_G(d_0)$ *defined in Lemma* 5.3 *satisfying* $d_{10} = d_2^y$.
(e) *There are* 4 *elements* l_1, l_2, l_3 *and* l_4 *in* $L - \Phi(L)$ *of orders* 9, 9, 9 *and* 3, *respectively, which generate* L, *and there are* 2 *elements* k *and* d *of orders* 8 *and* 3, *respectively, which generate* J *such that* $N_{10} = N_G(d_{10}) = \langle l_i, k, d \mid 1 \leqslant i \leqslant 4\rangle$ *has the following set* $\mathcal{R}(N_{10})$ *of defining relations*:

$$
\begin{aligned}
&l_1^9 = l_2^9 = l_3^9 = l_4^3 = l_1^{-1}l_2^{-1}l_1l_2 = l_2^{-1}l_1^3l_2^{-2} = 1,\\
&l_3l_1^3l_3^2 = l_3^{-1}l_1^{-1}l_3^{-1}l_1^{-1}l_3^{-1}l_1^{-1} = l_3^{-1}l_1l_3^{-1}l_1l_3^{-1}l_1 = 1,\\
&l_3^{-1}l_2^{-1}l_3^{-1}l_2l_3^{-1}l_2^{-1} = l_3^{-1}l_2l_3^{-1}l_2l_3^{-1}l_2 = 1,\\
&l_1^{-1}l_4l_3^{-1}l_1l_4^{-1}l_3 = l_4^{-1}l_2^{-1}l_4^{-1}l_2^{-1}l_4^{-1}l_2^{-1} = 1,\\
&l_3l_2l_3^{-1}l_1^{-2}l_3l_2^{-1}l_3^{-1}l_1^{-1} = l_1^{-1}l_3^{-1}l_4l_1^{-2}l_4l_3^{-2}l_4 = 1,\\
&l_1l_4^{-1}l_3^{-1}l_2^{-1}l_1^{-1}l_3l_4^{-1}l_2l_4^{-1} = l_2l_4^{-1}l_3l_2^{-1}l_1^{-1}l_3^{-1}l_4^{-1}l_1l_4^{-1} = 1,\\
&l_4l_1l_4l_2^{-1}l_1^{-1}l_4l_3l_2l_3^{-1} = d^3 = k^{-1}d^{-1}k^{-1}d^{-1} = 1,\\
&k^8 = kd^{-1}k^3d^{-1}k^2 = k^{-1}l_1^{-1}kl_2l_3l_4^{-1}l_3^{-1}l_1 = k^{-1}l_2^{-1}kl_3l_1l_4^{-1}l_1l_3l_1 = 1,\\
&k^{-1}l_3^{-1}kl_2l_3l_2^{-1}l_3^{-1}l_1^{-1}l_4^{-1} = k^{-1}l_4^{-1}kl_1^{-1}l_2^{-1}l_3^2l_1l_4^{-1} = 1,\\
&d^{-1}l_1^{-1}dl_1^{-1}l_4^{-1}l_2l_4^{-1}l_2l_4^{-1} = d^{-1}l_2^{-1}dl_1l_2l_3^{-1}l_4l_3l_2^{-1}l_4^{-1} = 1,\\
&d^{-1}l_3^{-1}dl_3^{-1}l_1^{-1}l_4l_3^{-1}l_2l_4^{-1} = d^{-1}l_4^{-1}dl_3l_2^{-1}l_4l_3^{-1}l_2^{-1} = 1.
\end{aligned}
$$

(f) N_{10} *has a faithful permutation representation of degree* 3^9 *with stabilizer* $J = \langle k, d\rangle$.
(g) *A system of representatives* n_i *of the* 62 *conjugacy classes* $n_i^{N_{10}}$ *of* $N_{10} = N_G(d_{10})$ *and the corresponding centralizer orders* $|C_{N_{10}}(n_i)|$ *are given in Table* 9.9.
(h) *The character table of* $N_{10} = N_G(d_{10})$ *is stated in Table* 10.9.

Proof. (a) By Lemma 5.6 $N_1 = N_G(D_1)$ has a faithful permutation representation of degree 3^9, and $S_1 = \langle M, d_3\rangle \leqslant N_1$ is a Sylow 3-subgroup of G with order $|S_1| = 3^{10}$. Applying MAGMA to this faithful permutation representation it is easy to check that the Frattini subgroup $\Phi(S_1)$ of S_1 has order 3^7.

(c) This statement is an immediate consequence of assertions (f), (g) and (h) of Lemma 5.1.

(b) By Lemma 5.1 and the Brauer–Suzuki theorem we know that

$$N_G(d_2)/O\big[N_G(d_2)\big] \cong N_H(d_2)/C \cong \mathrm{GL}_2(3). \tag{$*$}$$

As $|O[N_G(d_2)]| = |O[C_G(d_2)]| \leqslant 3^9$ by the proof of Lemma 5.5 and $d_2 = d_{10}^y$ for some involution $y \in N_0 = N_G(d_0)$ by Lemma 5.3, it follows that $|O[N_G(d_2)]| = 3^9$, because $Z(S_1) = \langle d_{10} \rangle$ and $|S_1| = 3^{10}$. Hence $L = O[N_G(d_{10})]$ has order 3^9.

By (a) $\bar{S}_1 = S_1/\Phi(S_1)$ is a 3-dimensional vector space over $F_3 = \mathbb{Z}/3\mathbb{Z}$. Thus each maximal subgroup of S_1 corresponds uniquely to a one-dimensional subspace of $\bar{S}_1$. Therefore there are exactly $\frac{3^3-1}{3-1} = 13$ maximal subgroups L_j, $1 \leqslant j \leqslant 13$, in S_1. One of them is $M = L_1$, say.

By (c) and Lemma 5.3 we know that $k_7 = z^y$, $J = J_1^y = \langle k_1, k_2, c_3, k_7 \rangle \leqslant N_{10} = N_G(d_{10})$ and $k_i^2 = z$ for $i = 1, 2$. Hence $V = \langle z, k_7 \rangle \leqslant N_G(L)$. Using MAGMA again it follows that there are only two maximal subgroups L_j of the 12 remaining maximal subgroups of S_1 which are normalized by V. If L_2 and L_3 denote these 2 maximal subgroups then another application of MAGMA yields that $|\operatorname{Aut}(L_2)| = 2^4 \cdot 3^{15}$ and $|\operatorname{Aut}(L_3)| = 2^2 \cdot 3^{12}$. As $d_{10} = d_2^y$ Eq. $(*)$ implies that $N_{10}/L \cong \mathrm{GL}_2(3)$. Hence $L = L_2$ is the unique maximal subgroup of S_1 which equals $O[N_G(d_{10})]$.

(d) As $J_1 \leqslant N_H(d_2) \leqslant N_G(d_2)$ and $d_{10} = d_2^y$ it follows that $J = J_1^y \leqslant N_{10} = N_G(d_{10}) = [N_G(d_2)]^y$. Since $J \cong \mathrm{GL}_2(3)$ does not have a proper normal subgroup of 3-power order we also get $J \cap L = 1$. Hence $N_{10} = JL$ by another application of $(*)$.

(e) The Frattini subgroup $\Phi(L)$ of $L = O(N_{10})$ has order 3^5 as has been checked by means of MAGMA. Hence L can be generated by 4 elements. The specific generators l_1, l_2, l_3 and l_4 have been determined by means of MAGMA and the faithful permutation representation of $N_1 = N_G(D_1)$ given in Lemma 5.6. Since $J \cong \mathrm{GL}_2(3)$ and $\langle k_9, d_3 \rangle \cong \mathrm{GL}_2(3)$ by the proof of Lemma 5.6 the subgroup J of N_{10} can be generated by an element k of order 8 and an element of order 3. Now (d) implies that $N_{10} = \langle l_i, k, d \mid 1 \leqslant i \leqslant 4 \rangle$ and that $\bar{L} = L/Z(L)$ and J can be identified with well determined subgroups of $\operatorname{Aut}(L) = \operatorname{Aut}(L_2)$ mentioned above. Using the algorithm of Cannon and Holt implemented in MAGMA the presentation of the split extension $N_{10} = LJ$ of L by J given in statement (e) has been calculated.

(f) This statement is a trivial consequence of (d).

(g) The representatives n_i of the 62 conjugacy classes of $N_{10} = N_G(d_{10})$ have been determined by means of algorithm [10], MAGMA and the faithful permutation representation of N_{10} given in (f).

(h) The character table of N_{10} has been obtained by application of MAGMA and the faithful permutation representation of N_{10}. This completes the proof. □

Proposition 5.8. *Let G be a finite simple group of* Th*-type with 2-central involution z such that $C_G(z) = H$, where H is described in Lemma* 2.2. *Let d_0, d_2, d_4 and d_{10} and the involution y be defined as in Lemmas* 5.1 *and* 5.3, *respectively. Then the following statements hold*:

(a) *G has* 3 *conjugacy classes of elements of order* 3 *represented by $3_A = d_0 \in H$, $3_B = d_2 = d_{10}^y \in H$ and $3_C = d_4 \in H$.*

(b) *G has nine 3_A-special conjugacy classes represented by the classes 3_a, 6_a, 12_a, 12_b, 21, 24_a, 24_b, 39_a, 39_b of $N_0 = N_G(d_0)$ classified in Table* 9.6.

Table 5.3

G	3_A	3_B	3_C
N_1	d, g	a, c, f, h, j	b, e, i, k
N_4	c, d, e, j	a, b, g, h, i, l, p	f, k, m, n, o, q
N_{10}	d, i	b, f, g, h, n, s, t	$a, c, e, j, k, l, m, o, p, q, r$
N_0	a, c, d, f, g	b, e, j	h, i

Table 5.4

G	9_A	9_B	9_C
N_1	a, c, g	b, d, e, f, h	i, j, k
N_{10}	a, c	b, d, h	e, f, g
N_0			a, b, c, d, e, f, g

(c) *G has fourteen* 3_B*-special conjugacy classes represented by the classes* 3_a, 6_a, 9_a, 9_b, 9_c, 12_a, 18_a, 18_b, 27_a, 27_b, 27_c, 36_a, 36_b, 36_c *of* $N_{10} = N_G(d_{10})$ *classified in Table* 9.9.

(d) *G has nine* 3_C*-special conjugacy classes represented by the classes* 3_a, 6_a, 12_a, 15_a, 15_b, 24_a, 24_b, 30_a, 30_b *of* $N_4 = N_G(d_4)$ *classified in Table* 9.7.

(e) *Using the classification of the* 3*-singular conjugacy classes of* $N_1 = N_G(D_1)$, $N_4 = N_G(d_4)$, $N_{10} = N_G(d_{10})$ *and* $N_0 = N_G(d_0)$ *given in Tables* 9.8, 9.7, 9.9 *and* 9.6, *respectively, the fusion of their conjugacy classes of elements of order* 3 *into the conjugacy classes of G is determined in Table* 5.3.

Deliberately, the element order part of the names of the fused classes has been omitted because it can be read off the particular first row.

(f) *G has three conjugacy classes of elements of order* 9. *Using the classification of the* 3*-singular conjugacy classes of* $N_1 = N_G(D_1)$, $N_{10} = N_G(d_{10})$ *and* $N_0 = N_G(d_0)$ *given in Tables* 9.8, 9.9 *and* 9.6, *respectively, the fusion of their conjugacy classes of elements of order* 9 *into the conjugacy classes of G is determined in Table* 5.4, *where the letters denote the indices of the classes of elements of order* 9 *given in Tables* 5.3 *and* 5.4.

(g) *G has three conjugacy classes* 27_A, 27_B *and* 27_C *of elements of order* 27 *represented by the three* 3_B*-special conjugacy classes* 27_a, 27_b *and* 27_c *of* N_{10} *given in Table* 9.9, *respectively. Also* $N_1 = N_G(D_1)$ *has three conjugacy classes* 27_a, 27_b *and* 27_c *by Table* 9.8. *They fuse in G to* 27_B, 27_A *and* 27_C, *respectively.*

(h) *Each finite simple group G of* Th*-type has* 32 3*-singular conjugacy classes.*

Proof. (a) Each finite simple group G of Th-type has at least 3 conjugacy classes of elements of order 3 by Lemma 5.1.

The 3-local subgroups $N_1 = N_G(D_1)$, $N_4 = N_G(d_4)$ and $N_{10} = N_G(d_{10})$ have 11, 20 and 17 conjugacy classes of elements of order 3 by Tables 9.8, 9.7 and 9.9, respectively. By Lemma 5.6 N_1 contains a Sylow 3-subgroup S_1 of G. Therefore it suffices to study the fusion of the 11 conjugacy classes of elements of order 3 belonging to N_1 in G.

Table 5.5

N_1	41	42	43	44	45	46	47	48	49	50	51
I	1, 2	6	3, 7	4, 5, 8	9, 13	11, 14	10, 15	12, 16, 20	17, 18, 19, 21	22	23

N_{10}	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
I	1	2, 3	5	4	8, 9	6, 10	7, 11	12, 22	13	15	14	16	17, 23	18	19	21	20

Table 5.6

N_1	41	42	43	44	48	49
J	53	52, 54, 56	60	55, 61	57, 58, 59, 67, 68, 72	62, 63, 64, 65, 66, 69, 70, 71, 73

Table 5.7

N_4	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93
J	52	53, 57	54, 65	55	56	59	60	58	61	63	70	62	69	67	73	66	64	72	68	71

By Lemmas 5.6 and 5.7 the intersection

$$I = N_1 \cap N_{10} = \langle S_1, z, k_7 \rangle = N_G(S_1).$$

Similarly, Lemmas 5.4 and 5.6 imply that

$$J = N_1 \cap N_4 = \langle S_4, k_9, k_{10} \rangle,$$

where S_4 is a Sylow 3-subgroup of N_4. Using now the faithful permutation representations of N_1, N_{10} and N_4 given in Lemmas 5.6, 5.7 and 5.4, respectively, and MAGMA the fusion of these classes has been analyzed as follows.

The group I has 23 conjugacy classes of elements of order 3. They are denoted by the numbers: $1, 2, \dots, 22, 23$. The 17 conjugacy classes of such 3-elements belonging to N_{10} are denoted by the numbers: $24, 25, \dots, 39, 40$. Finally, the 11 conjugacy classes of elements of order 3 of N_1 are denoted by: $41, 42, \dots, 50, 51$. Using now MAGMA the following fusion patterns have been calculated. See Table 5.5.

Therefore N_1 has the following 5 subsets of conjugacy classes which fuse in G:

$$\{41, 43, 46\}, \{42, 47\}, \{44, 45\}, \{48, 50\}, \{49, 51\}. \tag{$*$}$$

The group $J = N_1 \cap N_4$ has 22 conjugacy classes of elements of order 3. They are denoted by: $52, 53, \dots, 72, 73$. The 20 conjugacy classes of such 3-elements belonging to N_4 are denoted by: $74, 75, \dots, 92, 93$. Applying now the same methods as before to the triple N_1, J and N_4 one obtains the fusion patterns given in Table 5.6, the classes 45, 46, 47, 50 and 51 of N_1 have empty intersection with J. See Table 5.7.

Therefore N_1 has also the following 9 subsets of conjugacy classes which fuse in G:

$$\{41, 48\}, \{42, 49\}, \{43\}, \{44\}, \{45\}, \{46\}, \{47\}, \{50\}, \{51\}. \tag{$**$}$$

From ($*$) and ($**$) one obtains the following 3 connected components in the fusion graph:

$$\{41, 43, 46, 48, 50\}, \{42, 47, 49, 51\} \text{ and } \{44, 45\}.$$

Thus G has 3 conjugacy classes of elements of order 3. Their representatives are denoted by 3_B, 3_C and 3_A. Another application of Lemma 5.1 completes the proof of assertion (a).

(b) The 3_A-special conjugacy classes of G are classified by those 3-singular conjugacy classes of $N_0 = N_G(d_0)$ whose representatives power into $3_A = d_0$. Thus (b) follows immediately from Table 9.6.

The statements (c) and (d) follow now at once from Table 9.9 and 9.7, respectively.

(e) Using now the notations of Table 9.8 the conjugacy classes of elements of order 3 of the subgroup N_1 fusing to the conjugacy classes 3_A, 3_B and 3_C of G are the following

$$\begin{aligned}
3_A &\Leftrightarrow \{44, 45\} \Leftrightarrow \{d, g\},\\
3_B &\Leftrightarrow \{41, 43, 46, 48, 50\} \Leftrightarrow \{a, c, f, h, j\},\\
3_C &\Leftrightarrow \{42, 47, 49, 51\} \Leftrightarrow \{b, e, i, k\},
\end{aligned}$$

respectively.

The fusion patterns of the conjugacy classes of elements of order 3 of the subgroups N_{10} and N_4 into the classes 3_A, 3_B and 3_C have been determined similarly from the results obtained in the proof of (d).

In order to obtain the fusion of the conjugacy classes of elements of order 3 of $N_0 = N_G(d_0)$ in G, we consider the fusion of the conjugacy classes of $L = N_0 \cap N_1$ in the two subgroups N_0 and N_1 of G. Another application of MAGMA yields that L has 34 conjugacy classes of elements of order 3. They are denoted by the numbers: $1, 2, \ldots, 33, 34$.

In this argument the 11 conjugacy classes of elements of order 3 in N_1 are denoted by the numbers between 35 and 44, and the ones of N_0 by the numbers between 45 and 55. Applying now the methods explained in the proof of (d) to the triple N_1, L and N_0 one sees that N_1 and N_0 have the following seven subsets of conjugacy classes which fuse in G:

$$\{48, 49\}, \{45, 47, 50\}, \{46, 51\}, \{52\}, \{53\}, \{54\}, \{55\}, \qquad (***)$$

$$\{35, 37, 38, 40, 41\}, \{36, 39, 44\}, \{42, 43\}, \qquad (****)$$

respectively.

Using the notations of Table 9.6 the conjugacy classes of elements of order 3 of the subgroup N_0 fusing to the conjugacy classes 3_A, 3_B and 3_C of G are the following

$$\begin{aligned}
3_A &\Leftrightarrow \{35, 37, 38, 40, 41\} \Leftrightarrow \{a, c, d, f, g\},\\
3_B &\Leftrightarrow \{36, 39, 44\} \Leftrightarrow \{b, e, j\},\\
3_C &\Leftrightarrow \{42, 43\} \Leftrightarrow \{h, i\},
\end{aligned}$$

respectively.

(f) The 3-local subgroups $N_1 = N_G(D_1)$, $N_0 = N_G(d_0)$ and $N_{10} = N_G(d_{10})$ have 11, 6 and 8 conjugacy classes of elements of order 9 by Tables 9.8, 9.6 and 9.9, respectively. Using again the faithful permutation representations of N_1, N_{10} and N_0 given in Lemmas 5.6,

Table 5.8

N_1	17	18	19	20	21	22	23	24	25	26	27
I	1, 4	2, 7	3	6, 14	5	8	9	10	11, 13, 15	12	16

N_{10}	28	29	30	31	32	33	34	35
I	1	2, 6	3, 4, 9	5, 7, 8, 10	11, 12, 16	13	15	14

5.7 and 5.2, respectively, and MAGMA the fusion of these classes has been determined as follows.

The group $I = N_1 \cap N_{10}$ has 16 conjugacy classes of elements of order 9. They are denoted by the numbers: $1, 2, \ldots, 16$. The 11 conjugacy classes of such 9-elements belonging to N_1 are denoted by the numbers: $17, 18, \ldots, 27$. Finally, the 8 conjugacy classes of elements of order 9 of N_{10} are denoted by: $28, 29, \ldots, 35$. Using now MAGMA the fusion patterns, shown in Table 5.8, have been calculated.

Calculating the third powers of the representatives of these classes and using the fusion of the elements of order 3 given in (i) it follows that G has three conjugacy classes 9_A, 9_B and 9_C of elements of order 9 the representatives of which power to elements of order 3 lying in 3_A, 3_B and 3_C, respectively. Hence the conjugacy classes of elements of order 9 of N_1 fuse in G as follows:

$$\begin{aligned} 9_A &\Leftrightarrow \{17, 19, 23\} \Leftrightarrow \{a, c, g\}, \\ 9_B &\Leftrightarrow \{18, 20, 21, 22, 24\} \Leftrightarrow \{b, d, f, e, h\}, \\ 9_C &\Leftrightarrow \{25, 26, 27\} \Leftrightarrow \{i, j, k\}. \end{aligned}$$

(g) By Table 9.9 $N_{10} = N_G(d_{10})$ has exactly three conjugacy classes of elements of order 27 denoted by 27_a, 27_b and 27_c. Their 9th powers are conjugate to d_{10} by Table 10.9. Hence G has three 3_B-special conjugacy classes 27_A, 27_B and 27_C represented by 27_a, 27_b and 27_c of N_{10}. Also N_1 has 3 conjugacy classes 27_a, 27_b and 27_c of elements of order 27 by Table 9.8. Using again the fusion of the conjugacy classes of $I = N_1 \cap N_{10}$ in N_1 and N_{10} one obtains the fusion pattern given in the assertion.

(h) This final assertion follows immediately from the statements (a)–(d) and Theorem 2.2.11 of [14]. This completes the proof. □

6. The 5-, 7- and 13-singular conjugacy classes

Let G be a finite simple group of Th-type with a central involution z such that $C_G(z) = H$ defined in Lemma 2.2. Let π be the smallest set of prime divisors p of $|G|$ containing 2 such that $C_G(g)$ is a π-subgroup of G for each π-element $1 \neq g \in G$. By the results of the previous sections we know that π contains $\{2, 3, 5, 7, 13\}$. It is the purpose of this section to show that there are no further primes p in π. This is done by determination of the centralizers of the elements of prime order p of G, where $p \in \{5, 7, 13\}$. Most of the results presented here are due to D. Parrott [17]. New proofs are given for the sake of completeness.

Lemma 6.1. *Let G be a finite simple group of* Th*-type with a* 2*-central involution z such that $H = C_G(z) = \langle d, x, y\rangle$, where the generators x, y and d are defined in Lemma* 2.3. *Let $d_0 = x^8$. Then d_0 has order* 3, *and the following statements hold*:

(a) *$s_1 = xy$ is a representative of the unique conjugacy class of elements of order* 7 *in H.*
(b) *$N_0 = N_G(d_0)$ contains an element s_0 of order* 7 *and an involution k_6 such that $[k_6, s_0] = [s_0, d_0] = 1$.*
(c) *There are* 3 *elements q_1, q_2 and q_3 in H of orders* 8, 2 *and* 3, *respectively, such that $N_H(s_1) = \langle s_1, q_1, q_2, q_3\rangle$ has the following set of defining relations*:

$$s_1^7 = q_1^8 = q_2^2 = q_3^3 = 1, \qquad q_1^4 = z, \qquad q_1^{q_2} = q_1^{-1},$$
$$[q_1, q_3] = [q_2, q_3] = 1, \qquad s_1^{q_1} = s_1^{-1}, \qquad s_1^{q_2} = s_1, \qquad s_1^{q_3} = s_1^2.$$

(d) *$C_H(s_1) = \langle s_1\rangle \times \langle q_1^2, q_2\rangle$ and $D_8 = \langle q_1^2, q_2\rangle$ is a dihedral group of order* 8 *with center $Z(D_8) = \langle z\rangle$.*
(e) *There is a unique involution $q_4 \in \langle q_1, q_2\rangle$ commuting with q_2 and an element d_{12} of order* 3 *in the conjugacy class d_0^G such that $N_7 = N_G(s_1) = \langle s_1, q_2, q_3, q_4, d_{12}\rangle$ has the following set of defining relations*:

$$s_1^7 = q_2^2 = q_3^3 = q_4^2 = d_{12}^3 = 1, \qquad d_{12}^{q_4} = d_{12}^2,$$
$$[s_1, q_2] = [s_1, d_{12}] = 1, \qquad s_1^{q_4} = s_1^6, \qquad s_1^{q_3} = s_1^2,$$
$$[q_2, q_4] = [q_2, q_3] = [q_3, q_4] = [q_3, d_{12}] = 1,$$
$$(d_{12}q_2)^7 = 1 = \left(q_2 d_{12}^{-1} q_2 d_{12}\right)^4.$$

(f) *$N_7 = N_G(s_1) = (\langle s_1\rangle : \langle q_3\rangle \times \langle q_2, d_{12}\rangle) : \langle q_4\rangle$, and $C_G(s_1) = \langle s_1\rangle \times \langle q_2, d_{12}\rangle$, where $\langle q_2, d_{12}\rangle \cong \mathrm{PSL}_2(7)$.*
(g) *N_7 has a faithful permutation representation of degree* 882 *with stabilizer $\langle q_2, q_2^{d_{12}}\rangle$.*
(h) *A system of representatives n_i and the corresponding centralizer orders $|C_{N_7}(n_i)|$ of the* 33 *conjugacy classes $n_i^{N_7}$ of N_7 is given in Table* 9.7.
(i) *The character table of N_7 is stated in Table* 10.10.

Proof. (a) holds by Table 9.1.

(b) This assertion follows immediately from Lemma 5.2.

(c) The elements q_i, $i = 1, 2, 3$, of $N_H(s_1)$ in statement (c) have been found by means of the faithful permutation representation of $H = \langle z, c, d, e, s, t, u\rangle$ given in Lemma 2.2 and MAGMA. The following specific choice has been realized:

$$q_1 = t^{-1}u^{-1}set^{-1}desu^{-1}dus^{-2}edu^{-1},$$
$$q_2 = tseudcdeu^{-1}s^{-1}t^{-1},$$
$$q_3 = ts^{-1}t^{-1}usedcdedcds^{-1}tu.$$

Furthermore, it has been checked that these three elements q_i and s_1 generate $N_H(s_1)$ and satisfy the given relations.

(d) is an immediate consequence of (c).

(e) By (b) the element d_0 of order 3 belongs to $C_G(s_0)$. Let X be a Sylow 3-subgroup of $C_G(s_0)$ containing d_0. Suppose that $|X| > 3$. Then $|C_X(d_0)| \geqslant 9$. But

$$C_X(d_0) \leqslant C_G(s_0) \cap C_G(d_0) \leqslant C_{N_0}(s_0),$$

and $|C_{N0}(s_0)| = 42$ by Table 9.6. Hence $X = \langle d_0 \rangle$.

Since $[k_6, s_0] = 1 = [d_0, s_0]$ by (b) and $k_6^g = z$ for some $g \in G$ by Lemma 3.1 it follows from Table 9.1 that there is an element $h \in H = C_G(z)$ such that $s_0^{gh} = s_1 = xy$, and $d_{12} = d_0^{gh} \in C_G(s_1)$. In particular, $C_G(s_0)$ and $C_G(s_1)$ are conjugate subgroups of G. Thus $C_G(s_1)$ has a cyclic Sylow 3-subgroup of order 3. Its Sylow 2-subgroup is dihedral of order 8 by (d).

Let u be any involution of $C_G(s_1)$. Then $u^y = z$ for some $y \in G$ by Lemma 3.1, and $[z, s_1^y] = 1$. Hence $s_1^y \in H = C_G(z)$. By Table 9.1 H has a unique conjugacy class of elements of order 7. Thus $s_1^{yh_1} = s_1$ for some $h_1 \in H$. Therefore $u^{yh_1} = z^{h_1} = z$ and $yh_1 \in C_G(s_1)$. Hence $C_G(s_1)$ has a unique conjugacy class of involutions.

Let $Q = O[C_G(s_1)]$ and $V = \langle z, q_2 \rangle \leqslant D_8 \leqslant C_G(s_1)$. Another application of Table 9.1 yields that $C_Q(z) = \langle s_1 \rangle$. The Klein four group V operates on Q by conjugation. Hence

$$|Q|\left|C_Q(V)\right|^2 = \left|C_Q(z)\right|\left|C_Q(q_2)\right|\left|C_Q(q_2 z)\right| = \left|C_Q(z)\right|^3$$

by the Brauer–Wielandt theorem and the conjugacy of the 3 involutions q_2, $q_2 z$ and z of $C_G(s_1)$. As $|C_Q(V)| = |C_Q(z)| = |\langle s_1 \rangle| = 7$ we obtain that $Q = \langle s_1 \rangle$.

Therefore $L = C_G(s_1)/\langle s_1 \rangle$ is a simple group with a dihedral Sylow 2-subgroup D_8 of order 8 and a cyclic Sylow 3-subgroup of order 3. Thus $L \cong \mathrm{PSL}_2(7)$ by the Gorenstein–Walter theorem. Since the Schur multiplier of $\mathrm{PSL}_2(7)$ has order 2 it follows that $C_G(s_1) = \langle s_1 \rangle \times L$. Hence $L = \langle q_1^2, q_2, d_{12} \rangle \cong \mathrm{PSL}_2(7)$. Using the computational methods described in the proof of (c) it has been checked that $q_4 = tseu^{-1}dcdeus^{-1}t^{-1}$ is the only involution different from z and q_2 in the Sylow 2-subgroup $\langle q_1, q_2 \rangle$ of $N_G(s_1)$ which commutes with q_4. Furthermore, it does not belong to the Sylow 2-subgroup $\langle q_1^2, q_2 \rangle$ of L. Therefore it induces an outer automorphism of order 2 of L. Since $L \cong \mathrm{PSL}_2(7)$ has a unique conjugacy class of elements of order 3 it is easy to check by means of MAGMA that for each involution q of L there is an element w of order 3 in L such that

$$L = \left\langle q, w \mid q^2 = 1 = w^3,\ (qw)^7 = 1 = \left(qw^{-1}qw\right)^4 \right\rangle.$$

In fact, there are 336 such pairs (q, w) in L. As L is normalized by q_4 it follows that for $q = q_2$ there is an w of order 3 in L which is inverted by q_4 such that $L = \langle q_2, w \rangle$. As $d_{12} \in L$ is only uniquely determined up to G-conjugacy we may assume that $g \in G$ and $h \in H$ are chosen such that $d_{12} = d_0^{gh}$ is such an element w in L.

The other relations of $N_7 = N_G(s_1)$ of statement (e) have been deduced from (c) by means of MAGMA.

(f) This statement follows immediately from (e).

(g) Since $L = \langle q_2, d_{12}\rangle \cong \mathrm{PSL}_2(7)$ the involutions q_2 and $q_2^{d_{12}}$ do not commute. Hence $T = \langle q_2, q_2^{d_{12}}\rangle$ is a Sylow 2-subgroup of L which does not contain any normal subgroup of N_7. Thus (g) holds because $|N_7 : T| = 882$ by (f).

(h) The representatives n_i of the 33 conjugacy classes of $N_7 = N_G(s_1)$ have been determined by means of algorithm [10], the faithful permutation representation of N_7 given in (g) and MAGMA.

(i) The character table of N_7 has been obtained by application of MAGMA and the faithful permutation representation of N_7. This completes the proof. □

Lemma 6.2. *Let G be a finite simple group of* Th*-type. Then the following statements hold*:

(a) *G has elements s_1, q_2, q_3, q_4 and d_{12} of orders* 7, 2, 3, 2 *and* 3, *respectively, such that*

$$N_7 = N_G(s_1) = \bigl(\langle s_1\rangle : \langle q_3\rangle \times \langle q_2, d_{12}\rangle\bigr) : \langle q_4\rangle \quad \textit{and}$$

$$C_G(s_1) = \langle s_1\rangle \times \langle q_2, d_{12}\rangle, \quad \textit{where } L = \langle q_2, d_{12}\rangle \cong \mathrm{PSL}_2(7).$$

(b) *Let $s_2 = q_2 d_{12}$ and $q_5 = q_2 d_{12}^2 (q_2 d_{12})^2 (q_2 d_{12})^2 (q_2 d_{12})$ in L. Then $N_L(s_2) = \langle s_2\rangle : \langle q_5\rangle$ is a Frobenius group of order* 21, *and $[q_5, q_4] = 1$, $s_2^{q_4} = s_2^6$.*

(c) *$S_7 = \langle s_1, s_2\rangle$ is a Sylow* 7*-subgroup of G of order* 49, *and $C_G(S_7) = S_7$.*

(d) *Let $s_3 = s_1 s_2$, $q_6 = q_3 q_5$ and $q_7 = q_2 q_4$. Then one of the following* 2 *statements holds*:

  (d$_1$) *$s_1^G = s_3^G$, and G has one conjugacy class of elements of order* 7 *and $|N_G(S_7)| = 2^4 \cdot 3^2 \cdot 7^2$.*

  (d$_2$) *$s_1^G \neq s_3^G$, and G has two conjugacy classes of elements of order* 7. *Furthermore, $C_G(s_1) \cong \langle s_1\rangle \times \mathrm{PSL}_2(7)$, $C_G(s_3) = S_7$ and $N_G(s_3) = \langle s_1, s_2, q_6, q_7\rangle$ has the following set of defining relations*:

$$s_1^7 = s_2^7 = q_6^3 = q_7^2 = 1, \qquad [s_1, s_2] = 1,$$
$$[q_6, q_7] = 1, \qquad s_1^{q_6} = s_1^2, \qquad s_2^{q_6} = s_2^2,$$
$$s_1^{q_7} = s_1^6, \qquad s_2^{q_7} = s_2^6.$$

Proof. (a) holds by Lemma 6.1.

(b) By Lemma 6.1 the element s_2 of the simple group L has order 7, and it is inverted by the involution q_4 of $N_G(s_1)$. Its normalizer $N_L(s_2)$ is a Frobenius group of order 21, because $L = \langle s_2, d_2\rangle \cong \mathrm{PSL}_2(7)$. Using MAGMA the element q_5 in $N_L(s_2)$ of order 3 commuting with q_4 has been found computationally.

(c) Clearly $S_7 = \langle s_1, s_2\rangle$ is a Sylow 7-subgroup of $N_7 = N_G(s_1)$, and $|S_7| = 49$. From (a) and (b) follows that

$$N_{N_7}(S_7)/S_7 \cong \langle q_3\rangle \times \langle q_5\rangle \times \langle q_4\rangle,$$

and that N_7 has 4 conjugacy classes of elements of order 7 represented by s_1, s_2, s_1s_2 and $s_1s_2^{-1}$ with orbit lengths $|s_1^{N_7}| = 6 = |s_2^{N_7}|$, $|(s_1s_2)^{N_7}| = 18 = |(s_1s_2^{-1})^{N_7}|$. Hence

$$S_7 - \{1\} = s_1^{N_7} \cup s_2^{N_7} \cup (s_1s_2)^{N_7} \cup \left(s_1s_2^{-1}\right)^{N_7}. \tag{$*$}$$

As $L = \langle d_{12}, q_2\rangle$ and $[q_3, q_2] = [q_3, d_{12}] = 1$ by Lemma 6.1 it follows that $[q_3, s_2] = 1$. Since $C_G(d_4)$ and $C_G(d_{10})$ have coprime orders to 7 by Tables 9.7 and 9.9, respectively, it follows from Proposition 5.8 that q_3 is conjugate to d_0 in G. Therefore s_1 and s_2 are conjugate in G by Table 9.10. Hence $|C_G(s_1)| = |C_G(s_2)|$.

Let $N_2 = N_G(S_7)$. Since $C_G(s_1) = \langle s_1\rangle \times L$ and $C_L(s_2) = \langle s_2\rangle$ we know that $C_G(S_7) = S_7$. Therefore $\bar{N}_2 = N_G(S_7)/S_7$ is isomorphic to a subgroup of $\mathrm{GL}_2(7)$ which has a cyclic Sylow 7-subgroup of order 7.

Suppose that S_7 is not a Sylow 7-subgroup of G. Then 7^3 divides $|N_2|$ but not $|C_G(s_1)| = |C_G(s_2)|$. Hence the length of each orbit $s_1^{N_2}$ and $s_2^{N_2}$ in $S_7 - \{1\}$ is 42, which is impossible by $(*)$. Thus (c) holds.

(d) Since S_7 is an abelian Sylow 7-subgroup of G, and the elements $s_1, s_2 \in S_7$ are conjugate in G. Burnside's lemma implies that $s_1^{N_2} = s_2^{N_2}$. As $|N_2/S_7|$ is coprime to 7 theorem of Schur and Zassenhaus states that S_7 has a complement $K \cong N_2/S_7$ in N_2 which is uniquely determined up to conjugation in N_2. Therefore we may assume that $K \leqslant \mathrm{GL}_2(7)$. It is well known that the singer cycle of this general linear group has order 48. But G does not have elements of order 16 by Lemma 3.1. By the proof of (c) we may assume that $U = \langle q_3\rangle \times \langle q_5\rangle \times \langle q_4\rangle$ is a subgroup of K. As $s_1^{N_2} = s_2^{N_2}$ there is an element $k \in K - U$, and $|K| = 3^2 \cdot 2^a$, where $2 \leqslant a \leqslant 4$. Using MAGMA and the matrices of $q_3, q_5, q_4 \in \mathrm{GL}_2(7)$ with respect to the basis $\{s_1, s_2\}$ of the GF(7)-vector space S_7 it has been checked that either $|K| = 3^2 \cdot 2^2$ and $N_2 = S_7K$ has 2 orbits $s_1^{N_2}$ and $(s_1s_2)^{N_2}$ on $S_7 - \{1\}$ or $|K| = 3^2 \cdot 2^4$ and N_2 acts transitively on $S_7 - \{1\}$ by conjugation.

Let $s_3 = s_1s_2$. Then the assertion of statement (d_1) holds now by another application of Burnside's lemma.

(d_2) Suppose that $s_1^G \neq s_3^G$. Therefore $s_1^{N_2} \neq s_3^{N_2}$ and $|K| = 2^2 \cdot 3^2$. Let $L_1 = C_G(s_3)$. Then S_7 is an abelian Sylow 7-subgroup of L_1 and $N_3 = N_{L_1}(S_7) = N_2 \cap L_1$. Using the precise group structure of $N_2/S_7 = K \leqslant \mathrm{GL}_2(7)$ and its conjugate action on S_7 it follows that $N_3 = C_{L_1}(S_7) = S_7$.

Therefore Burnside's theorem states that S_7 has a normal complement B_1 in L_1. The involution q_7 of $N_G(s_1)$ inverts s_3 and it commutes with the element q_6 of order 3 which satisfies $s_3^{q_6} = s_3^2$ by (b) and Lemma 6.1. Hence $q_7 \in N_G(s_3) \geqslant L_1 = C_G(s_3)$ operates on B_1 by conjugation. Since s_1 and s_3 are not conjugate in G the involution q_7 operates fixed point freely on B_1 by Lemma 3.1 and Table 9.1 Lemma 6.1 states that $C_G(s_1) = \langle s_1\rangle \times L$, where $L \cong \mathrm{PSL}_2(7)$. Hence $s_1 \in S_7$ operates fixed point freely on B_1 as well. Since $\langle s_1, q_7\rangle$ is a dihedral group of order 14, Theorem 8.3.2 of [11] implies that $B_1 = 1$. Hence $L_1 = C_G(s_3) = S_7$. As $N_G(s_3)/L_1 = \langle q_7, q_6\rangle$ is cyclic of order 6 the other assertions of (d_2) follow now at once from statements (a) and (b). This completes the proof. □

Lemma 6.3. *Let G be a finite simple group of* Th*-type with a* 2*-central involution z such that $H = C_G(z) = \langle d, x, y\rangle$, where the generators x, y and d are defined in Lemma* 2.3. *Then the following statements hold*:

(a) $e = x^2y^2 \in H$ *has order* 30, $f = e^6$ *is a representative of the unique conjugacy class of elements of order* 5 *in* H, *and* $d_{13} = e^{10} \in H$ *has order* 3.

(b) $C_H(f)$ *contains two elements* w_1, w_2 *of order* 4 *such that* $Q_8 = \langle w_1, w_2 \mid w_1^{w_2} = w_1^{-1}, w_1^2 = z = w_2^2\rangle$ *is a quaternion group of order* 8 *which is normalized by* d_{13}, *and* $C_H(f) = \langle f\rangle \times [Q_8 : \langle d_{13}\rangle]$.

(c) $N_H(f)$ *contains an element* w *of order* 8 *such that* $f^w = f^2$, $w_1^w = w_1^{-1}$, $w_2^w = w_1w_2^{-1}$, $d_{13}^w = d_{13}^2$ *and* $w^4 = z$.

(d) $v = w^2w_1$ *is an involution, and* v *and* z *are conjugate in* $N_H(f)$.

(e) *Either* $C_G(f) = C_H(f)$ *or* $C_G(f) = QC_H(f)$, *where* $Q = O[C_G(f)]$ *is an extra-special* 5*-group of order* 125 *and exponent* 5 *which is also a Sylow* 5*-subgroup of* G.

(f) *If* $C_G(f) \neq C_H(f)$ *then* $N_5 = N_G(f) = \langle f, f_1, f_2, w_1, w_2, w, d_{13}\rangle$ *has the following set of defining relations*:

$$
\begin{gathered}
f_1^5 = f_2^5 = f^5 = [f_1, f] = [f_2, f] = w^8 = w_1^4 = 1,\\
w_2^4 = d_{13}^3 = 1, \qquad f_1f_2^{-1}f_1^{-1}f_2f_1^{-1} = 1,\\
w^4 = w_1^2 = w_2^2 = z, \qquad w_1^{w_2} = w_1^{-1},\\
d_{13}^w = d_{13}^2, \qquad w_1^{-1}w^{-1}w_1^{-1}w = 1,\\
w_1^{-1}d_{13}^{-1}w_2d_{13} = 1 = d_{13}w^{-1}d_{13}w,\\
w^{-1}w_2^{-1}w_1^{-1}ww_2^{-1} = 1,\\
f_1^{w_1} = f_2f_1^{-1}f_2, \qquad f_2^{w_1} = f_2f_1^{-1}f_2^{-2},\\
f^{w_1} = f, \qquad f_1^{w_2} = f_1f_2f_1,\\
f_2^{w_2}f_2^3, \qquad f^{w_2} = f,\\
f_1^w = f_2f_1^{-2}f_2, \qquad f_2^w = f_2f_1^{-1}f_2,\\
f^w = f^{-2}, \qquad f_1^{d_{13}} = f_2, \qquad f^{d_{13}} = f,\\
f_2^{d_{13}} = f^2f_2^{-1}f_1^{-1}.
\end{gathered}
$$

(g) *If* $C_G(f) \neq C_H(f)$, *then* $N_5 = N_G(f)$ *has* 27 *conjugacy classes* $n_i^{N_5}$ *whose representatives* n_i *and centralizer orders* $|C_{N_5}(n_i)|$ *are given in Table* 9.11, *and the character table of* N_5 *is stated in Table* 10.14. *Furthermore,* N_5 *has a faithful permutation representation of degree* 125 *with stabilizer* $\langle w_1, w_2, w, d_{13}\rangle$.

(h) *In any case,* G *has a unique conjugacy class of elements of order* 5.

Proof. Assertion (a) follows immediately from Tables 9.3 and 9.1 classifying the conjugacy classes of the groups $D = \langle x, y\rangle$ and $H = \langle x, y, d\rangle$, respectively.

Statements (b), (c) and (d) have been checked computationally using the faithful permutation representation of H given in Lemma 2.2 and MAGMA.

(e) Suppose that $C_G(f) \neq C_H(f)$. Since Q_8 is a Sylow 2-subgroup of $C_G(f)$ by (b), the theorem of Brauer and Suzuki asserts that

$$C_G(f) = QC_H(f), \quad \text{where } Q = O\bigl[C_G(f)\bigr]. \tag{$*$}$$

By Proposition 5.8 G has 3 conjugacy classes of elements of order 3. They are represented by d_0, d_4 and $d_2 = d_{10}^y \in H$. By Lemmas 5.2 and 5.7 the prime 5 does not divide the orders of $N_0 = N_G(d_0)$ and $N_{10} = N_G(d_{10})$. Furthermore, Lemma 5.4 states that $N_4 = N_G(d_4)$ has order $|N_4| = 2^5 \cdot 7^3 \cdot 5$, $N_4/O[N_4] \cong 2S_6$, and $|O(N_4)| = 3^5$. Therefore 3 does not divide the order of $Q = O[C_G(f)]$.

By Lemma 6.2 the centralizers of elements of order 7 in G have orders which are coprime to 5. Hence $(|Q|, 7) = 1$.

Let $v \in N_H(f)$ be the involution defined in (d). Then $V = \langle z, v\rangle$ is a Klein four subgroup of H which operates on Q by conjugation. The theorem of Brauer and Wielandt states that

$$|Q|\bigl|Cq(V)\bigr|^2 = \bigl|C_Q(z)\bigr|\bigl|C_Q(v)\bigr|\bigl|C_Q(vz)\bigr|. \tag{$**$}$$

Now $C_Q(V) = Q \cap C_G(z) \cap C_G(v) = Q \cap C_H(v) = 1$ by (d), because $(3, |Q|) = 1$. By Lemma 3.1 G has one conjugacy class of involutions. Since 5 and 7 divide $|H|$ to the first power only, and $(|Q|, 7) = 1$ it follows from $(**)$ that $|Q| \leqslant 5^3$. As $C_G(f) \neq C_H(f)$ and the involutions v and vz are conjugate in $N_H(f)$ Eq. $(**)$ implies that $|Q| = 5^3$, $N_G(f) = QN_H(f)$ by (c).

Let $C_Q(v) = C_G(v) \cap Q = \langle f_1\rangle$. Then $f_1 \in f^G$ by Table 9.1 and Lemma 3.1. However, $f_1 \notin \langle f\rangle$ because $C_Q(V) = 1$.

Suppose that Q is abelian. Then it follows from (c) that $Q - \{1\}$ has 3 $N_H(f)$-orbits with representatives f, f_1, and ff_1 of lengths 4, 24 and 96, respectively. Hence $N_5 = N_G(f) = N_G(Q) = QN_H(f)$, and Q is an abelian Sylow 5-subgroup of G. As $f_1 \in f^G$ and $f_1, f \in Q$ Burnside's lemma implies that $f_1^{N_5} = f^{N_5} = \langle f\rangle$, a contradiction to $f_1 \notin \langle f\rangle$.

Therefore Q is not abelian. Its center $Z(Q) = \langle f\rangle$. Hence $N_G(Q) \leqslant N_G(f) = QN_H(f)$, and Q is a Sylow 5-subgroup of G. As $f_1 \notin Z(Q)$, it has 120 conjugates in $N_G(f)$. Therefore Q has exponent 5, which shows that (e) holds.

(f) If $C_G(f) = C_H(f)$, then $\langle f\rangle$ is a Sylow 5-subgroup of G, and f^G is the unique conjugacy class of elements of order 5 by Table 9.1 and Sylow's theorem.

Suppose that $C_G(f) \neq C_H(f)$. By the theorem of Schur and Zassenhaus the Sylow 5-subgroup $\langle f\rangle$ of $N_H(f)$ has a complement which is uniquely determined up to conjugation. By (c) the subgroup $L_2 = \langle w_1, w_2, w, d_{13}\rangle$ is such a complement. Hence $N_G(f) = QN_H(f) = QL_2$, and $Q \cap L_2 = 1$. Another application of the theorem of Schur and Zassenhaus shows that the split extension $N_G(f)$ of Q by L_2 is uniquely determined up to isomorphism.

Using now the algorithm of Cannon and Holt implemented in MAGMA it is easy to construct this split extension $N_5 = N_G(f)$ explicitly by means of generators and relations.

The presentation of N_5 given in statement (f) has been obtained computationally by these methods.

(g) and (h): Clearly, N_5 has a faithful permutation representation of degree 125 with stabilizer L_2. Using it and MAGMA the character table of N_5 has been calculated. In particular, N_5 has a unique conjugacy class of elements of order 5 by Table 8.8.9.

The representatives n_i of the 27 conjugacy classes of $N_5 = N_G(f)$ have been determined by means of Kratzer's algorithm [10], MAGMA and the faithful permutation representation of N_5. This completes the proof. □

Lemma 6.4. *Let G be a finite simple group of* Th*-type with 2-central involution z such that $H = C_G(z) = \langle x, y, d\rangle$, where the generators x, y, d are defined in Lemma* 2.3. *Let $d_0 = x^8$. Then the following statements hold*:

(a) *$N_0 = N_G(d_0)$ has a cyclic Sylow 13-subgroup $\langle u\rangle$ of order 13 such that $C_{N_0}(u) = \langle d_0\rangle \times \langle u\rangle$.*
(b) *$U = C_G(u)$ has a cyclic Sylow 3-subgroup $\langle d_0\rangle$ of order 3.*
(c) *$U = C_G(u) = \langle d_0\rangle \times \langle u\rangle$.*
(d) *There is an element $w \in N_0$ of order 12 such that $N_{13} = N_G(u) = \langle d_0, u, w\rangle$ has the following set of defining relations*:

$$d_0^3 = u^{13} = w^{12} = [d_0, u] = 1, \qquad u^w = u^7, \qquad d_0^w = d_0^2.$$

(e) *G has three 13_A-special conjugacy classes represented by the classes 13_a, 39_a, 39_b of $N_0 = N_G(d_0)$ given in Table* 9.6.

Proof. (a) follows immediately from Lemma 5.2 and Table 9.6.

(b) Let X be a Sylow 3-subgroup of $U = C_G(u)$ containing $\langle d_0\rangle$. Suppose that $|X| \geqslant 9$. Then $|C_X(d_0)| \geqslant 9$. But (a) implies that

$$C_X(d_0) \leqslant C_G(u) \cap C_G(d_0) \leqslant C_{N_0}(u) = \langle d_0\rangle \times \langle u\rangle,$$

a contradiction. Thus (b) holds.

(c) By Lemma 3.1 G has a unique conjugacy class of involutions. As $(|H|, 13) = 1$, the order of U is odd. Hence 3 is the smallest prime dividing $|U|$. Therefore the Sylow 3-subgroup $\langle d_0\rangle$ of U has a normal 3-complement K in U by Corollary 1.3.18 of [14]. Thus K is a characteristic subgroup of U and $N_{13} = N_G(u)$. By the character Table 9.6 of $N_0 = N_G(d_0)$ we know that $N_{N_0}(u)/C_{N_0}(u)$ is cyclic of order 12. Now (a) and the theorem of Schur and Zassenhaus imply the existence of an element $w \in N_{N_0}(u)$ of order 12 such that $N_{N_0}(u) = \langle d_0, u, w\rangle$ has the set of defining relations given in assertion (d). In particular, statement (d) holds if the proof of (c) can be completed.

As U has odd order we have $U = O(N_{13})$, and $N_{13} = U : \langle w\rangle$. Hence $D = \langle d_0, w^4\rangle$ is a Sylow 3-subgroup of N_{13} which is elementary abelian of order 9. It operates on the characteristic subgroup K of U by conjugation. As $(|K|, |D|) = 1$ Theorem 8.3.4 of [11, p. 172], states that $K = \langle C_K(d) \mid 1 \neq d \in D\rangle$. Let $\pi = \{2, 3, 5, 7, 13\}$. Then $C_G(d)$

is a π-group for each $d \in D$ by Lemmas 5.2, 5.4 and 5.7. As K is of odd order and centralized by the 13-element u it follows from Lemmas 6.1–6.3 that K is a 13-group. Now Lemma 5.2 implies that $C_K(d_0) = \langle u \rangle$.

Let $d_{11} := w^4$. Then $d_0^w = d_0^2$ implies that $(d_0 d_{11})^w = d_0^2 d_{11} \in N_0$. By Proposition 5.8 G has exactly 3 conjugacy classes of elements of order 3 represented by the 3 different conjugacy classes d_0^H, d_4^H and d_2^H. All their elements are real by the character table of H, see Table 9.1. Therefore all products $d_0^k d_{11}^j$ with $k, j \in \{1, 2\}$ are conjugate in G to their inverses. Using the faithful permutation representation of N_0 given in Lemma 5.2 and MAGMA it has been checked that there is an element $n \in N_0$ which commutes with d_0 such that $d_{11}^n = d_4$. Thus $(d_0 d_{11})^n = d_0 d_4$. Another application of MAGMA and the faithful permutation representation of H described in Lemma 2.2 yields that $d_0 d_4 \in d_2^H$. Therefore d_{11}, d_{11}^2 and all products $d_0^k d_{11}^j$ with $k, j \in \{1, 2\}$ are not conjugate to d_0 in G by Lemma 5.1. Now Proposition 5.8 and Tables 9.7 and 9.9 imply that $C_G(d) \cap K = 1$ for all $1 \neq d \in D$ different from d_0 and d_0^2. Hence $K = C_K(d_0) = \langle u \rangle$, and $U = \langle u \rangle \times \langle d_0 \rangle$.

(e) By (d) $N_{13} = N_G(u)$ is a subgroup of N_0. Therefore this assertion follows immediately from (a), Sylow's theorem and the power map information given in Table 10.6. This completes the proof. □

Proposition 6.5. *Let G be a finite simple group of* Th*-type with a* 2*-central involution z such that $C_G(z) = H$. Let $\pi = \{2, 3, 5, 7, 13\}$. Then π is the smallest set of prime divisors p of $|G|$ such that the centralizer $C_G(x)$ of any π-element x of G is a π-subgroup of G.*

Proof. By Theorem 2.2.11 in [14] it suffices to show the assertion for elements $x \in G$ of prime order $p \in \pi$.

Lemma 3.1 states that G has a unique conjugacy class of involutions. Hence the result is true for $p = 2$ by Lemma 2.2.

By Proposition 5.8 G has 3 conjugacy classes of elements of order 3, and they are represented by d_0, d_4 and d_{10}. Furthermore, $C_G(x)$ is a π-subgroup of G for each $x \in \{d_0, d_4, d_{10}\}$ by Lemmas 5.2, 5.4 and 5.7.

G has a unique conjugacy class f^G of elements of order 5 and $C_G(f)$ is a π-subgroup of G by Lemma 6.3.

For $p = 7$ two cases may occur by Lemma 6.2. But this result also states that $C_G(x)$ is a π-subgroup of G for each element $x \in G$ of order 7 in any case. Quoting now Lemma 6.4 completes the proof. □

7. Group order

In this section we show that all finite simple groups G of Th-type have the same group order

$$|G| = 2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31,$$

Table 7.1

r_i	$\lvert C_G(r_i)\rvert$	$d(r_i)$	r_i	$\lvert C_G(r_i)\rvert$	$d(r_i)$
2_A	$2^{15}\cdot 3^4\cdot 5\cdot 7$	30,510	9_C	$2\cdot 3^4$	54
3_A	$2^6\cdot 3^7\cdot 7\cdot 13$	8424	10_A	$2^3\cdot 3\cdot 5$	30
3_B	$2^3\cdot 3^{10}$	2916	12_C	$2^2\cdot 3^3$	36
3_C	$2^4\cdot 3^7\cdot 5$	810	12_D	$2^3\cdot 3$	12
4_A	$2^{11}\cdot 3^3\cdot 7$	2160	13_A	$3\cdot 13$	13
4_B	$2^9\cdot 3\cdot 5$	240	14_A	$2^3\cdot 7$	28
5_A	$2^3\cdot 3\cdot 5^3$	150	18_A	$2^3\cdot 3^2$	36
6_A	$2^4\cdot 3^3\cdot 5$	90	18_B	$2\cdot 3^2$	18
6_B	$2^6\cdot 3^3$	72	20_A	$2^2\cdot 5$	20
6_C	$2^3\cdot 3^4$	108	21_A	$3\cdot 7$	21
7_A	$2^3\cdot 3\cdot 7^2$	196	27_A	3^3	27
8_A	$2^7\cdot 3$	32	28_A	$2^2\cdot 7$	28
9_A	$2^3\cdot 3^6$	324	36_B	$2^2\cdot 3^2$	36
9_B	3^6	81			

and one conjugacy class of elements of order 7. This completes the classification of all p-singular conjugacy classes of G for all primes p in $\pi = \{2, 3, 5, 7, 13\}$ by the results of the previous sections.

By Proposition 6.5 the set π is the smallest set of prime divisors p of $|G|$ such that the centralizer $C_G(x)$ of any π-element x of G is a π-subgroup of G. Let $g_{\pi'} = |G|/g_\pi$ where π' is the set of all primes q which do not belong to π. Let t be the number of strongly real π'-conjugacy classes of G. In order to show that the invariant t of G is computable from the given presentation of $H = C_G(z)$ we need the following subsidiary result.

Lemma 7.1. *Let G be any finite simple group of* Th*-type with a* 2*-central involution z such that $C_G(z) = H$. Let $\pi = \{2, 3, 5, 7, 13\}$. Then exactly one of the following assertions holds*:

(a) *The Sylow* 5*-subgroups of G have order* 125*, G has a unique conjugacy class of elements of order* 7*, G has* 27 *strongly real conjugacy classes r_i^G, $1 \leqslant i \leqslant 27$, of π-elements $r_i \neq 1$ and their invariants $d(r_i) = |\{(u, v) \in z^G \times z^G \mid uv = r_i\}|$ and centralizer orders $|C_G(r_i)|$ are given in Table* 7.1.
Furthermore, $c = 1 + \sum_{i=1}^{27} d(r_i)\frac{|H|}{|C_G(r_i)|} = 3^4 \cdot 5 \cdot 7 \cdot 17 \cdot 18{,}341$.

(b) *The Sylow* 5*-subgroups of G have order* 5*, G has a unique conjugacy class of elements of order* 7 *and* 27 *strongly real conjugacy classes r_i^G of π-elements r_i, and $c_2 = 1 + \sum_{i=1}^{27} d(r_i)\frac{|H|}{|C_G(r_i)|} = 3^4 \cdot 7 \cdot 1{,}591{,}753$.*

(c) *The Sylow* 5*-subgroups of G have order* 125*, G has two conjugacy classes of elements of order* 7 *and* 28 *strongly real conjugacy classes r_i^G of π-elements r_i, and $c_3 = 1 + \sum_{i=1}^{28} d(r_i)\frac{|H|}{|C_G(r_i)|} = 3^6 \cdot 5^2 \cdot 7 \cdot 13 \cdot 19 \cdot 31$.*

(d) *The Sylow* 5*-subgroups of G have order* 5*, G has two conjugacy classes of elements of order* 7 *and* 28 *strongly real conjugacy classes r_i^G of π-elements r_i, and $c_4 = 1 + \sum_{i=1}^{28} d(r_i)\frac{|H|}{|C_G(r_i)|} = 3^4 \cdot 7^2 \cdot 250{,}799$.*

Proof. By the results of the previous sections we know that each simple group G of Th-type has one conjugacy class $2_A = z^G$ of involutions, three classes $3_A = d_0^G$, $3_B = d_4^G$, $3_C = d_{10}^G$ of elements of order 3, one class $5_A = f^G$ of elements of order 5 and one class $13_A = u^G$ of elements of order 13. Furthermore, Lemma 6.2 states that G has either one conjugacy class $7_A = s_1^G$ or two classes $7_A = s_1^G$ and $7_B = s_3^G$ of elements of order 7, and that in the latter case $N_G(s_3) \leqslant N_G(s_1)$. By Lemma 6.3 a Sylow 5-subgroup of G has either order 5 or 125. Therefore there are exactly 4 cases for the distribution of the strongly real conjugacy classes of π-elements of G. For any simple group G of Th-type the character tables of $H = C_G(2_A)$ and the normalizers $N_0 = N_G(3_A)$, $N_4 = N_G(3_B)$, $N_{10} = N_G(3_C)$, $N_5 = N_G(5_A)$, and $N_7 = N_G(7_A)$ are given in Section 10.

Let R be any of these groups. By Lemma 3.1 G has only one conjugacy class z^G of involutions. Suppose that $z_1, z_2, \ldots, z_k$ are representatives of the conjugacy classes of involutions of R. For each irreducible character ψ of R let

$$\psi(\hat{R}) = \sum_{i=1}^{k} \bigl|R : C_R(z_i)\bigr| \psi(z_i).$$

Then by Lemma 8.2.1 in [14] for each strongly real π-element r of G with extended centralizer $C_G^*(r) \leqslant R$ the following equation holds:

$$d(r) = \bigl|\bigl\{(x, y) \in z^G \times z^G \mid xy = r\bigr\}\bigr| = \frac{1}{|R|} \cdot \sum_{\psi \in \mathrm{Irr}_{\mathbb{C}}(R)} \psi(\hat{R})^2 \psi\bigl(r^{-1}\bigr)\psi(1)^{-1}.$$

In each case this formula has been applied to all normalizers of representatives of the conjugacy classes y^G of elements of prime order p in $\pi = \{2, 3, 5, 7, 13\}$.

(a) In this case a complete classification of all conjugacy classes of π-elements is known. The orders of the centralizers $C_G(r_i)$ can be read off from the power map information in the character tables given in Section 10. Applying the above formulas to these character tables we obtain the invariants $d(r_i)$ in the table of (a). The invariant c is then trivially calculated by means of the entries in this table.

(b) Lemma 6.3 states now that $C_G(f) = C_H(f)$ and that again G has only one conjugacy class f^G of elements of order 5. Using the character table of the group H it follows that the invariant $d(5_A) = d(f) = 30$ and that $|C_G(f)| = 120$. All other entries of the table of (b) remain unchanged. Hence the assertion follows.

(c) In this case $C_G(f) \neq C_H(f)$, but G has two conjugacy classes of order 7 represented by s_1 and s_3, and $N_G(s_3) \leqslant N_G(s_1)$ by Lemma 6.2. Hence $|d(s_3) = d(7_B)|$ can be calculated from Table 10.10 and the above stated formula. It follows that $d(7_B) = 49 = |C_G(7_B)|$ and that $c_3 = c + |H| = 976841775$.

(d) In this final case $c_4 = c_2 + |H|$, which is now easily computed. This completes the proof. □

Proposition 7.2. *Let G be any finite simple group of* Th*-type with a* 2*-central involution z such that $C_G(z) = H$. Let $\pi = \{2, 3, 5, 7, 13\}$. Then the following assertions hold*:

(a) *G has one conjugacy class of elements of order* 7 *and* 27 *strongly real conjugacy classes r_i^G*, $1 \leqslant i \leqslant 27$, *of π-elements $r_i \neq 1$.*

(b) *G has order*

$$|G| = 2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31.$$

Proof. In order to bound the number t of strongly real π'-conjugacy classes of G we check the conditions of Theorem 4.3.9 of [14]. By Proposition 6.5 $\pi = \{2, 3, 5, 7, 13\}$ is the smallest set of prime divisors p of G such that $C_G(x)$ is a π-subgroup of G for each π-element x of G. Hence 11 is the smallest prime not contained in π. Lemma 7.1 states that in each of its 4 cases the number s of strongly real conjugacy classes of π-elements is at most 28 and that $g_\pi/|H|$ is greater than s. Let c be any of the constants c or c_i with $i = 2, 3, 4$ defined in Lemma 7.1. It has been checked that the maximal integral solution k of the inequality $11^{y-1} < c|H|g_\pi^{-1} \cdot 11^{-1} + \frac{1}{2}|H|^2 g_\pi^{-1} y$ is $k = 3$ and $k = 4$ in the cases (a), (c) and (b), (d), respectively. Since $H = C_G(z)$ has maximal order among all the centralizers of the elements of G in any finite simple group G of Th-type Theorem 4.3.9(b) from [14] asserts that G has at most $3|H|$ and $4|H|$ different strongly real conjugacy classes of π'-elements in the cases (a), (c) and (b), (d), respectively.

By Lemma 3.1 any finite simple group G of Th-type has a unique conjugacy class of involutions. Therefore $m = |G : H| = c + t|H|$ by Theorem 4.3.8(d) of [14]. As $E = 2^5 L_5(2)$ is a subgroup of G by Lemma 3.1 the prime 31 is a divisor of $|E|$. Hence 31 divides $c + t|H|$ in any of the 4 cases of Lemma 7.1. Furthermore,

$$|G| = g_\pi \cdot g_{\pi'} = |H| \cdot \bigl(c + t|H|\bigr).$$

Therefore the integers 18,135 and 3627 divide $c + t|H|$ in the cases (a), (c) and (b), (d) of Lemma 7.1, respectively. Using now the bounds for the number t of strongly real π'-conjugacy classes established above and Lemma 7.1 it follows that $x = c + t|H|$ is of the form

$$x_a(i) = 2835\bigl(311{,}797 + 2^{15}(1 + 18{,}135i)\bigr), \quad 0 \leqslant i \leqslant 15{,}366,$$

$$x_b(i) = 567\bigl(1{,}591{,}753 + 163{,}840(2177 + 3627i)\bigr), \quad 0 \leqslant i \leqslant 51{,}225,$$

$$x_c(i) = 2835\bigl(344{,}565 + 2^{15}(18{,}135i)\bigr), \quad 0 \leqslant i \leqslant 15{,}366,$$

$$x_d(i) = 567\bigl(1{,}755{,}593 + 163{,}840(2176 + 3627i)\bigr), \quad 0 \leqslant i \leqslant 51{,}225$$

in the cases (a), (b), (c) and (d), respectively. By Theorem 4.5.2 of [14] in each of all these cases the factor $g_{\pi'}$ of G has to satisfy the Frobenius congruence

$$1 + g_{\pi'}\left[\sum_{i=1}^{9} \frac{g_\pi}{c(x_i)}\right] \equiv 0 \mod 3^{10}, \tag{$*$}$$

where $c(x_i)$ are the centralizer orders of the 9 conjugacy classes of 3-power order: 3_A, 3_B, 3_C, 9_A, 9_B, 9_C, 27_A, 27_B and 27_C. From the character tables of the 3-normalizers of G given in Section 10 follows immediately that

$$F_1 = \sum_{i=1}^{9} \frac{g_\pi}{c(x_i)} = 2^9 \cdot 5^2 \cdot 7 \cdot 19 \cdot 829 \cdot 12{,}973$$

in the cases (a) and (c), and that

$$F_2 = \sum_{i=1}^{9} \frac{g_\pi}{c(x_i)} = 2^9 \cdot 7 \cdot 19 \cdot 829 \cdot 12{,}973$$

in the cases (b) and (d) of Lemma 7.1, respectively. The proof is now completed case by case.

(a) For each integer $0 \leqslant i \leqslant 15{,}366$ let $y_a(i) = |H|x_a(i)g_\pi^{-1}$. Evaluating the Frobenius congruence $1 + F_1 \cdot y_a(i) \equiv 0 \bmod 3^{10}$ yields that it is only satisfied for $i = 0$. Hence $m = c + |H|$ and $t = 1$.

(c) Now let $y_c(i) = |H|x_c(i)g_\pi^{-1}$ for $0 \leqslant i \leqslant 15{,}366$. It follows that in this case the congruence $1 + F_1 \cdot y_c(i) \equiv 0 \bmod 3^{10}$ has only the solution $i = 0$. Hence $m = c_3 = 3^6 \cdot 5^2 \cdot 7 \cdot 13 \cdot 19 \cdot 31$, and

$$|G| = |H|c_3 = 2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31.$$

Now Sylow's theorem implies that the Sylow 19-normalizer $N_{19} = N_G(v)$ of G has order $19 \cdot 18$, where v is an element of order 19 in G. Hence N_{19} is a Frobenius group. Therefore

$$d(v) = \frac{1}{|N_{19}|} \sum_{\psi \in \mathrm{Irr}_{\mathbb{C}}(N_{19})} \psi(\hat{N}_{19})^2 \psi\bigl(v^{-1}\bigr)\psi(1)^{-1} = 19$$

by the character table of the Frobenius group N_{19}. Hence v is a strongly real π'-element of order 19, and $t > 0$. Thus Theorem 4.3.8(d) of [14] implies that $|G| = |H| \cdot (c_2 + t|H|) \neq |H| \cdot c_2$. This contradiction shows that case (c) of Lemma 7.1 does not occur.

(b) Let $y_b(i) = |H|x_b(i)g_\pi^{-1}$ for $0 \leqslant i \leqslant 51{,}225$. Evaluating the Frobenius congruence $1 + F_2 \cdot y_b(i) \equiv 0 \bmod 3^{10}$ yields that it is only satisfied for $i = 3936$, $10{,}497$, $17{,}058$, $30{,}180$, $36{,}741$, $43{,}302$, besides the two additional solutions $i = 23{,}617$, $49{,}863$ for which the number $y_b(i)$ is divisible by 7 or 13. But both primes do not belong to π'. Therefore the 2 additional solutions can be neglected. The six special cases are now dismissed by application of Theorem 4.5.2 of [14] for the prime $p = 7$. Lemma 6.2 states that G has a unique conjugacy class s_1^G of elements of order 7 and a Sylow 7-subgroup of order 49. Furthermore, $|C_G(s_1)| = 1176$ by Table 9.10. Hence the corresponding Frobenius constant $F_3 = 5{,}240{,}401{,}920$. Therefore $1 + F_3 \cdot y_b(i) \equiv 5, 5, 40, 18, 7, 45 \bmod 49$ for $i = 3936$, $10{,}497$, $17{,}058$, $30{,}180$, $36{,}741$, $43{,}302$, respectively. Thus case (b) of Lemma 7.1 does not occur.

(d) Let $y_d(i) = |H|x_d(i)g_\pi^{-1}$ for $0 \leqslant i \leqslant 51{,}225$. Evaluating the Frobenius congruence $1 + F_2 \cdot y_d(i) \equiv 0 \bmod 3^{10}$ yields that it is only satisfied for $i = 3936$, 10,497, 17,058, 30,180, 36,741, 43,302, besides the two additional solutions $i = 23{,}619$, 49,863 for which the number $y_b(i)$ is divisible by 7 or 13. But both primes do not belong to π'. Therefore the 2 additional solutions can be neglected. The six special cases are now dismissed by application of Theorem 4.5.2 of [14] for the prime $p = 7$. Lemma 6.2 states that G has two conjugacy classes s_1^G and s_2^G of elements of order 7 and a Sylow 7-subgroup of order 49. Furthermore, $|C_G(s_1)| = 1176$ and $|C_G(s_2)| = 49$ by Table 9.10 and Lemma 6.2. Hence the corresponding Frobenius constant $F_4 = 131{,}010{,}048{,}000$. Therefore $1 + F_4 \cdot y_d(i) \equiv 7, 26, 45, 34, 4, 23 \bmod 49$ for $i = 3936$, 10,497, 17,058, 30,180, 36,741, 43,302, respectively. Thus case (d) of Lemma 7.1 does not occur. This completes the proof. □

Corollary 7.3. *Let G be any finite simple group of* Th*-type with a* 2*-central involution z such that $C_G(z) = H$. Then the following assertions hold*:

(a) *G has* 48 *conjugacy classes.*
(b) *G has one strongly real conjugacy π'-class*; *its elements have order* 19.
(c) *The Sylow* 19*-normalizer is a Frobenius group of order* $19 \cdot 18$.
(d) *The Sylow* 31*-normalizer is a Frobenius group of order* $31 \cdot 15$.
(e) *G has two nonreal conjugacy classes of elements of order* 31.

Proof. All statements are immediate consequences of Lemma 3.1, Proposition 5.8, Lemmas 6.3, 6.4, Proposition 7.2, Sylow's theorem and the class equation of G. □

8. Uniqueness proof and concrete character table

In this section we show that each finite simple group G of Th-type has a uniquely determined ordinary irreducible character of degree 248. Hence all conditions of the uniqueness criterion stated in [13] have been satisfied. It implies that each finite simple group G of Th-type is isomorphic to the simple group $\mathfrak{G}$ constructed in Theorem 4.2. Furthermore, a complete set of representatives of the 48 conjugacy classes of $\mathfrak{G}$ is given, each of which is a short word in the 4 generators of $\mathfrak{G}$. The character table of $\mathfrak{G}$ has been computed as well.

Proposition 8.1. *Let G be a finite simple group of* Th*-type. Then the minimal degree of a faithful irreducible complex character χ of G is* 248*, and G has exactly one character $\chi \in \operatorname{Irr}_{\mathbb{C}}(G)$ of this degree.*

Proof. Let $H = C_G(z)$ be the centralizer of a 2-central involution of G. Let S be the Sylow 2-subgroup of H defined in Lemma 2.2, and let A be the uniquely determined maximal elementary abelian normal subgroup of S. Let $D = N_H(A)$ and $E = N_G(A)$. Then $G = \langle H, E\rangle$ by Corollary 4.8.3 of [14]. Now Lemma 3.1 states that the amalgam $H \leftarrow D \rightarrow E$ has Goldschmidt index 1. Therefore the free product $P = H *_D E$ of H and E with amalgamated subgroup D is uniquely determined up to isomorphism. By the proof of Theorem 4.2 the smallest degree of a compatible pair $(\varphi, \tau) \in mf\,\mathrm{char}(H) \times mf\,\mathrm{char}(E)$ is 248, and there is exactly one such pair. As G is a simple epimorphic image of P it follows that G has at most one faithful $\chi \in \operatorname{Irr}_{\mathbb{C}}(G)$ with minimal degree 248.

Table 8.1

i	1_A	2_A	3_A	3_B	3_C	4_A	4_B	5_A	6_A	6_B	6_C
$\chi(g_i)$	248	−8	14	5	−4	8	0	−2	4	−2	1

i	7_A	8_A	8_B	9_A	9_B	9_C	10_A	12_A	12_B	12_C	12_D
$\chi(g_i)$	3	0	0	5	−4	2	2	2	2	−1	0

i	13_A	14_A	15_A	15_B	18_A	18_B	19_A	20_A	21_A
$\chi(g_i)$	1	−1	1	1	1	−2	1	0	0

i	24_A	24_B	24_C	24_D	27_A	27_B	27_C	28_A	30_A	30_B
$\chi(g_i)$	0	0	0	0	2	−1	−1	1	−1	−1

i	31_A	31_B	36_A	36_B	36_C	39_A	39_B
$\chi(g_i)$	0	0	−1	−1	−1	1	1

Using Brauer's characterization of characters we now prove that any finite simple group G of Th-type has an irreducible character $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$ of degree 248. In order to do so we define a class function $\chi : G \to \mathbb{C}$ by Table 8.1.

Let p be any prime divisor of $|G|$. If Y is an over-group of a p-elementary subgroup X, and the restriction $\chi|_y$ of X to Y is a generalized character, then also $\chi|_X$ is a generalized character of X. By Corollary 7.3 the Sylow 19- and 31-normalizers of G are Frobenius groups of orders $19 \cdot 18$ and $31 \cdot 15$, respectively. Using their character tables and the fusion of the elements of orders 18 and 15 in G it is easy to check that the restrictions of χ to these normalizers N_{19} and N_{31_A}, N_{31_B} are equivalent to the character of the sum of a linear character and 13 indecomposable projective characters and to the character of the sum of 8 indecomposable projective characters, respectively. Furthermore, $N_G(13_A)$ needs not be considered because it is conjugate to a subgroup of $N_0 = N_G(d_0) = N_G(3_A)$ by Lemma 6.4. Therefore by Corollary 2.8.10 of [14] it remains to show that $X|_Y$ is a generalized character of Y, whenever Y belongs to the following set of subgroups

$$\mathfrak{Y} = \{H, N_0, N_4, N_{10}, N_5, N_7\}$$

of G.

In the previous sections the fusion patterns of the conjugacy classes of the groups $Y \in \mathfrak{Y}$ have been determined. Therefore the inner products $(\chi, \xi)_Y$ can be calculated for each $\xi \in \mathrm{Irr}_{\mathbb{C}}(Y)$ and each $Y \in \mathfrak{Y}$ by means of the character tables given in Section 10. It turns out that:

$$\chi|_H = \chi_{14} + \chi_{18}, \quad \text{where } \chi_{14}, \chi_{18} \in \mathrm{Irr}_{\mathbb{C}}(H),$$

$$\chi|_{N_0} = \chi_3 + \chi_{13} + \chi_{19}, \quad \text{where } \chi_3, \chi_{13}, \chi_{19} \in \mathrm{Irr}_{\mathbb{C}}(N_0),$$

$$\chi|_{N_4} = \chi_{13} + \chi_{32} + \chi_{40}, \quad \text{where } \chi_{13}, \chi_{32}, \chi_{40} \in \mathrm{Irr}_{\mathbb{C}}(N_4),$$

$$\chi|_{N_{10}} = \chi_{10} + \chi_{14} + \chi_{47} + \chi_{50}, \quad \text{where } \chi_i \in \mathrm{Irr}_{\mathbb{C}}(N_{10}),$$

$$\chi|_{N_5} = \chi_{19} + \chi_{21} + \chi_{22} + \chi_{24} + \chi_{25} + \chi_{26} + \chi_{27}, \quad \text{where } \chi_i \in \mathrm{Irr}_{\mathbb{C}}(N_5),$$

$$\chi|_{N_7} = \chi_4 + \chi_6 + \chi_7 + \chi_{13} + \chi_{18} + \chi_{22} + 2\chi_{26} + \chi_{29} + \chi_{30} + \chi_{31} + 2\chi_{32} + \chi_{33},$$
$$\text{where } \chi_i \in \mathrm{Irr}_{\mathbb{C}}(N_7),$$

and that the inner product $(\chi, \chi)_G = 1$. Therefore χ is an irreducible character of G with $\chi(1) = 248$ by Corollary 2.8.10 of [14]. This completes the proof. □

Theorem 8.2. *Let H be the finite group of even order defined in Lemma* 2.2. *Then each finite simple group G of* Th*-type is isomorphic to the finite simple group $\mathfrak{G} \leqslant \mathrm{GL}_{248}(11)$ of order $|\mathfrak{G}| = 2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$ described in Theorem* 4.2.

Proof. By Theorem 4.2 there exists a finite simple group $\mathfrak{G}$ of order $|\mathfrak{G}| = 2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$ which is of Th-type.

Let G be any finite simple group of Th-type. By Lemma 3.1 G has a unique conjugacy class z^G of involutions. Hence we may assume that $C_G(z) = H$, where H is the group of even order defined in Lemma 2.2. It also asserts that the Sylow 2-subgroup S of H contains a uniquely determined maximal elementary abelian normal subgroup A. Let $D = N_H(A)$. By Lemma 3.1 $E = N_G(A)$ is uniquely determined by H up to isomorphism. Furthermore, $G = \langle H, E\rangle$ by Corollary 4.8.3 of [14]. Now Lemma 3.1 states that the amalgam $H \leftarrow D \rightarrow E$ has Goldschmidt index 1, and that $\mathfrak{G}$ has a unique 11-modular irreducible representation over GF(11) of minimal degree 248.

By Proposition 7.2 any finite simple group G of Th-type has order $|G| = |\mathfrak{G}|$. In particular 11 does not divide $|G|$. Therefore Maschke's theorem and Proposition 8.1 imply that each such group G has a unique irreducible 11-modular representation of degree 248. Hence $G \cong \mathfrak{G}$ by [13]. This completes the proof. □

Theorem 8.3. *Let $\mathfrak{G} = \langle \mathfrak{d}, \mathfrak{x}, \mathfrak{y}, \mathfrak{a}\rangle \leqslant \mathrm{GL}(248, 11)$ be the Thompson group constructed in Theorem* 4.2. *Then the following statements hold*:

(a) *$\mathfrak{G}$ has* 48 *conjugacy classes with representatives given in Table* 8.2.
(b) *The complex character table of the finite simple group $\mathfrak{G}$ coincides with the one of* Th *given in the Atlas* [4, p. 176].

Proof. (a) By Theorem 4.2 the matrix group $\mathfrak{G}$ is a simple group of Th-type. Since all simple groups G of Th-type are isomorphic by Theorem 8.2 it follows from Corollary 7.3 that $\mathfrak{G}$ has 48 conjugacy classes. Furthermore, by the proof of 8.2 there is an explicit isomorphism between $G = \langle d, x, y, a\rangle$ and $\mathfrak{G} = \langle \mathfrak{d}, \mathfrak{x}, \mathfrak{y}, \mathfrak{a}\rangle \leqslant \mathrm{GL}(248, 11)$ given by mapping the generators of G to their corresponding matrices in $\mathfrak{G}$. Thus both groups may be identified.

For $p \neq 19$ all p-singular conjugacy classes of $\mathfrak{G}$ have representatives in $H = \langle d, x, y\rangle$, $E = \langle a, x, y\rangle$, N_0, N_4, N_{10}, N_5 and N_7 by the classification of the conjugacy classes of any simple group G of Th-type given in the previous sections. All such representatives belonging to H or E are short words in the 4 generators of $\mathfrak{G}$. Therefore they are included in Table 8.2. Representatives of the other classes are given in the tables of Section 9 in terms of words in the 4 generators of $\mathfrak{G}$. Using their matrix properties like traces, determinants

Table 8.2

i	$\mathfrak{g}_i$	$C_{\mathfrak{G}}(\mathfrak{g}_i)$	i	$\mathfrak{g}_i$	$C_{\mathfrak{G}}(\mathfrak{g}_i)$
1_A	1	$\lvert\mathfrak{G}\rvert$	15_A	$(\mathfrak{d}\mathfrak{y}\mathfrak{x})^2$	$2\cdot 3\cdot 5$
2_A	$\mathfrak{x}^{12}$	$2^{15}\cdot 3^4\cdot 5\cdot 7$	15_B	$(\mathfrak{d}\mathfrak{y}\mathfrak{x})^{14}$	$2\cdot 3\cdot 5$
3_A	$\mathfrak{x}^8$	$2^6\cdot 3^7\cdot 7\cdot 13$	18_A	$(\mathfrak{d}\mathfrak{x}\mathfrak{y}\mathfrak{x})^2$	$2^3\cdot 3^2$
3_B	$(\mathfrak{d}\mathfrak{x})^3$	$2^3\cdot 3^{10}\cdot 7\cdot 13$	18_B	$\mathfrak{d}\mathfrak{x}\mathfrak{y}^2x$	$2\cdot 3^2$
3_C	$\mathfrak{y}$	$2^4\cdot 3^7\cdot 5$	19_A	$\mathfrak{a}\mathfrak{x}\mathfrak{y}\mathfrak{d}$	19
4_A	$\mathfrak{x}^6$	$2^{11}\cdot 3^3\cdot 7$	20_A	$\mathfrak{d}\mathfrak{y}$	$2^2\cdot 5$
4_B	$(\mathfrak{d}\mathfrak{y})^5$	$2^9\cdot 3\cdot 5$	21_A	$\mathfrak{a}\mathfrak{y}$	$3\cdot 7$
5_A	$(\mathfrak{d}\mathfrak{y})^4$	$2^3\cdot 3\cdot 5^3$	24_A	$\mathfrak{x}$	$2^3\cdot 3$
6_A	$(\mathfrak{d}\mathfrak{y}\mathfrak{x})^5$	$2^4\cdot 3^3\cdot 5$	24_B	$\mathfrak{x}^5$	$2^3\cdot 3$
6_B	$\mathfrak{x}^4$	$2^6\cdot 3^3$	24_C	$(\mathfrak{d}\mathfrak{x}\mathfrak{d}\mathfrak{x}^3)^{13}$	$2^3\cdot 3$
6_C	$(\mathfrak{d}\mathfrak{x}\mathfrak{y}\mathfrak{x})^6$	$2^3\cdot 3^4$	24_D	$\mathfrak{d}\mathfrak{x}\mathfrak{d}\mathfrak{x}^3$	$2^3\cdot 3$
7_A	$\mathfrak{x}\mathfrak{y}$	$2^3\cdot 3\cdot 7^2$	27_A	$\mathfrak{a}\mathfrak{d}\mathfrak{x}\mathfrak{y}\mathfrak{x}$	3^3
8_A	$\mathfrak{x}^3$	$2^7\cdot 3$	27_B	$\mathfrak{a}\mathfrak{d}\mathfrak{x}$	3^3
8_B	$\mathfrak{d}\mathfrak{x}\mathfrak{y}\mathfrak{d}\mathfrak{y}$	$2^5\cdot 3$	27_C	$(\mathfrak{a}\mathfrak{d}\mathfrak{x})^2$	3^3
9_A	$(\mathfrak{d}\mathfrak{x}\mathfrak{y}\mathfrak{x})^4$	$2^3\cdot 3^6$	28_A	$\mathfrak{d}\mathfrak{x}^2$	$2^2\cdot 7$
9_B	$(\mathfrak{a}\mathfrak{d}\mathfrak{x})^3$	3^6	30_A	$\mathfrak{d}\mathfrak{y}\mathfrak{x}$	$2\cdot 3\cdot 5$
9_C	$\mathfrak{d}\mathfrak{x}$	$2\cdot 3^4$	30_B	$(\mathfrak{d}\mathfrak{y}\mathfrak{x})^7$	$2\cdot 3\cdot 5$
10_A	$(\mathfrak{d}\mathfrak{y})^2$	$2^3\cdot 3\cdot 5$	31_A	$\mathfrak{a}\mathfrak{y}\mathfrak{x}^2$	31
12_A	$\mathfrak{x}^2$	$2^5\cdot 3^2$	31_B	$(\mathfrak{a}\mathfrak{y}\mathfrak{x}^2)^3$	31
12_B	$\mathfrak{x}^{10}$	$2^5\cdot 3^2$	36_A	$\mathfrak{d}\mathfrak{y}^2\mathfrak{x}$	$2^2\cdot 3^2$
12_C	$(\mathfrak{d}\mathfrak{x}\mathfrak{y}\mathfrak{x})^3$	$2^2\cdot 3^3$	36_B	$(\mathfrak{d}\mathfrak{y}^2\mathfrak{x})^5$	$2^2\cdot 3^2$
12_D	$\mathfrak{x}^3\mathfrak{y}$	$2^3\cdot 3$	36_C	$\mathfrak{d}\mathfrak{x}\mathfrak{y}\mathfrak{x}$	$2^2\cdot 3^2$
13_A	$\mathfrak{a}\mathfrak{y}^2\mathfrak{d}\mathfrak{y}$	$3\cdot 13$	39_A	$\mathfrak{a}\mathfrak{d}\mathfrak{a}\mathfrak{x}$	$3\cdot 13$
14_A	$\mathfrak{x}\mathfrak{y}^2$	$2^3\cdot 7$	39_B	$(\mathfrak{a}\mathfrak{d}\mathfrak{a}\mathfrak{x})^7$	$3\cdot 13$

or rational canonical forms their corresponding representatives given in the above table have been determined by means of Kratzer's algorithm [10]. The matrix representative of the strongly real conjugacy class 19_A has been found by forming random words in the 4 generators of $\mathfrak{G}$.

(b) Let $T = \{\mathfrak{g}_r \in \mathfrak{G} \mid 1 \leqslant r \leqslant 143{,}127{,}000\}$ be a right transversal of the stabilizer $\mathfrak{T}$ of the faithful permutation representation of $\mathfrak{G}$ constructed in Theorem 4.2. Let $\mathfrak{G} = \bigcup_{k=1}^{11} \mathfrak{T}r_k\mathfrak{T}$ be the double coset decomposition described in statement (γ) of that section. To each double coset $\mathfrak{T}r_k\mathfrak{T}$ belongs an intersection matrix $M_k = (d_{ij}^{(k)}) \in \mathrm{Mat}_{11}(\mathbb{Z})$, where $d_{ij}^{(k)} = |\mathfrak{T}|^{-1}|\mathfrak{T}r_i^{-1}\mathfrak{T}r_j \cap \mathfrak{T}r_k\mathfrak{T}|$. It follows from the first author's explicit calculations with the large permutations corresponding to the matrix generators of $\mathfrak{G}$ that the intersection algebra $\mathfrak{B} = \langle M_k \mid 1 \leqslant k \leqslant 11\rangle$ is commutative and that it is generated by the 3 intersection matrices M_2, M_3 and M_5 stated in Section 11. The first author has also computed the Gollan–Ostermann numbers $m_{jk} = |\{\mathfrak{g}_r \in T \mid \mathfrak{T}\mathfrak{g}_r y_j \mathfrak{g}_r^{-1} \in \mathfrak{T}r_k\mathfrak{T}\}|$, $1 \leqslant j \leqslant 48$, $1 \leqslant k \leqslant 11$, for all conjugacy classes $(y_j)^{\mathfrak{G}}$ and double coset representatives r_k by means of the permutation representation of $\mathfrak{G}$ mentioned above. They are stated in Table 11.5.

Since all intersection matrices M_k have a very special first row vector whose entries are all 0 except at the kth place where it is 1 one can easily reconstruct all 11 intersection matrices from the 3 given ones. Their eleven common eigenvectors f_i are stated in Table 11.4.

Let $\psi_i \in \mathrm{Irr}_{\mathbb{C}}(\mathfrak{G})$ be the irreducible constituent of $(1_{\mathfrak{T}})^{\mathfrak{G}}$ corresponding to the eigenvector f_i for $1 \leqslant i \leqslant 11$. Let $\{\mathfrak{y}_j \mid 1 \leqslant j \leqslant 48\}$ be the set of representatives of the conjugacy classes $\mathfrak{y}_j^{\mathfrak{G}}$ and let m_{jk} be the corresponding Gollan–Ostermann numbers, respectively. Let the eigenvalues c_{ki} be defined by $M_k \cdot f_i = f_i \cdot c_{ki}$. Then Corollary 5.2 of [15] asserts:

$$\psi_i(1) = |\mathfrak{G} : \mathfrak{T}| \left(\sum_{k=1}^{11} \left|\mathfrak{T} \cap \mathfrak{T}^{r_k}\right|^{-1} c_{ki}\bar{c}_{ki} \right)^{-1},$$

and

$$\psi_i(\mathfrak{y}_j) = \frac{\psi_i(1)}{|\mathfrak{G}|} \sum_{k=1}^{5} c_{ki} \left|\mathfrak{T} \cap \mathfrak{T}^{r_k}\right| m_{jk}.$$

Inserting the data stored in Section 11 into these formulas yields all the character values of the 11 nonequivalent irreducible constituents ψ_i of the permutation representation $(1_{\mathfrak{T}})^{\mathfrak{G}}$. They are stated in Table 11.6. In Proposition 8.1 the unique irreducible character of degree 248 of $\mathfrak{G}$ has been given explicitly. Since we also have the complete character tables of all normalizers of elements of prime order and the fusion of their conjugacy classes in $\mathfrak{G}$ one can easily complete the character table of $\mathfrak{G}$. □

9. Representatives of conjugacy classes

Table 9.1
Conjugacy classes of $H = \langle d, x, y \rangle$

i	h_i	$\lvert C_H(h_i)\rvert$
1_a	1	$2^{15} \cdot 3^4 \cdot 5 \cdot 7$
2_a	x^{12}	$2^{15} \cdot 3^4 \cdot 5 \cdot 7$
2_b	$(xy^2)^7$	$2^{14} \cdot 3 \cdot 7$
2_c	d	$2^{10} \cdot 3$
3_a	y	$2^4 \cdot 3^3 \cdot 5$
3_b	x^8	$2^6 \cdot 3^3$
3_c	$(dx)^3$	$2^3 \cdot 3^4$
4_a	x^6	$2^{11} \cdot 3^3 \cdot 7$
4_b	$(dy)^5$	$2^9 \cdot 3 \cdot 5$
4_c	$(dxdydy)^2$	$2^{10} \cdot 3$
4_d	$(dy^2x^2)^2$	2^9
4_e	$(dydyxy)^2$	2^9
4_f	$xyxy^2$	2^6
5_a	$(dy)^4$	$2^3 \cdot 3 \cdot 5$
6_a	$(dyx)^5$	$2^4 \cdot 3^3 \cdot 5$
6_b	x^4	$2^6 \cdot 3^3$
6_c	$(dxyx)^6$	$2^3 \cdot 3^4$
6_d	$(dx^3dy)^2$	$2^5 \cdot 3$
6_e	dx^2y	$2^3 \cdot 3$
7_a	xy	$2^3 \cdot 7$
8_a	x^3	$2^7 \cdot 3$
8_b	$dxydy$	$2^5 \cdot 3$
8_c	$dxdydy$	2^6
8_d	dy^2x^2	2^5
8_e	$dydyxy$	2^5
9_a	$(dxyx)^4$	$2^3 \cdot 3^2$
9_b	dx	$2 \cdot 3^2$
10_a	$(dy)^2$	$2^3 \cdot 3 \cdot 5$
12_a	x^2	$2^5 \cdot 3^2$
12_b	x^{10}	$2^5 \cdot 3^2$
12_c	$(dxyx)^3$	$2^2 \cdot 3^3$
12_d	dx^3dy	$2^4 \cdot 3$
12_e	$(dx^3dy)^5$	$2^4 \cdot 3$
12_f	x^3y	$2^3 \cdot 3$
14_a	$(dx^2)^2$	$2^3 \cdot 7$
14_b	$(xy^2)^3$	$2^2 \cdot 7$
14_c	xy^2	$2^2 \cdot 7$
15_a	$(dyx)^2$	$2 \cdot 3 \cdot 5$
15_b	$(dyx)^{14}$	$2 \cdot 3 \cdot 5$
18_a	$(dxyx)^2$	$2^3 \cdot 3^2$
18_b	dxy^2x	$2 \cdot 3^2$
20_a	dy	$2^2 \cdot 5$
24_a	x	$2^3 \cdot 3$
24_b	x^5	$2^3 \cdot 3$
24_c	$(dxdx^3)^{13}$	$2^3 \cdot 3$
24_d	$dxdx^3$	$2^3 \cdot 3$
28_a	dx^2	$2^2 \cdot 7$
30_a	dyx	$2 \cdot 3 \cdot 5$
30_b	$(dyx)^7$	$2 \cdot 3 \cdot 5$
36_a	dy^2x	$2^2 \cdot 3^2$
36_b	$(dy^2x)^5$	$2^2 \cdot 3^2$
36_c	$dxyx$	$2^2 \cdot 3^2$

Table 9.2
Conjugacy classes of $E = \langle x, y, a\rangle$

i	m_i	$\lvert C_M(m_i)\rvert$
1_a	1	$2^{15}\cdot 3^2\cdot 5\cdot 7\cdot 31$
2_a	x^{12}	$2^{15}\cdot 3^2\cdot 5\cdot 7$
2_b	a	$2^{11}\cdot 3\cdot 7$
3_a	x^8	$2^6\cdot 3^2\cdot 7$
3_b	y	$2^3\cdot 3^2\cdot 5$
4_a	x^6	$2^{11}\cdot 3\cdot 7$
4_b	$(ayx)^3$	$2^9\cdot 3$
4_c	$(ax^2)^2$	2^7
5_a	$(ax)^2$	$2\cdot 3\cdot 5$
6_a	x^4	$2^6\cdot 3^2$
6_b	$(ayx)^2$	$2^3\cdot 3^2\cdot 5$
6_c	$axax^4$	$2^4\cdot 3$
7_a	$(ay)^9$	$2^3\cdot 3\cdot 7$
7_b	$(ay)^3$	$2^3\cdot 3\cdot 7$
8_a	x^3	$2^7\cdot 3$
8_b	ay^2x	2^5
8_c	ax^2	2^4
10_a	ax	$2\cdot 3\cdot 5$
12_a	x^{10}	$2^5\cdot 3$
12_b	x^2	$2^5\cdot 3$
12_c	ayx	$2^2\cdot 3$
14_a	axy	$2^3\cdot 7$
14_b	$(axy)^3$	$2^3\cdot 7$
14_c	axy^2	$2^2\cdot 7$
14_d	$(axy^2)^3$	$2^2\cdot 7$
15_a	$(x^2y)^7$	$2\cdot 3\cdot 5$
15_b	x^2y	$2\cdot 3\cdot 5$
21_a	ay	$3\cdot 7$
21_b	$(ay)^5$	$3\cdot 7$
24_a	x^5	$2^3\cdot 3$
24_b	x	$2^3\cdot 3$
28_a	$(axax^2y)^3$	$2^2\cdot 7$
28_b	$axax^2y$	$2^2\cdot 7$
30_a	$(x^2y^2)^7$	$2\cdot 3\cdot 5$
30_b	x^2y^2	$2\cdot 3\cdot 5$
31_a	$(ayx^2)^5$	31
31_b	$(ayx^2)^{15}$	31
31_c	ayx^2	31
31_d	$(ayx^2)^{11}$	31
31_e	$(ayx^2)^3$	31
31_f	$(ayx^2)^7$	31

Table 9.3
Conjugacy classes of $D = \langle x, y\rangle$

i	d_i	$\lvert C_D(d_i)\rvert$	$\uparrow H$	$\uparrow M$
1_a	1	$2^{15}\cdot 3^2\cdot 5\cdot 7$	1_a	1_a
2_a	x^{12}	$2^{15}\cdot 3^2\cdot 5\cdot 7$	2_a	2_a
2_b	$(xy^2)^7$	$2^{14}\cdot 3\cdot 7$	2_b	2_a
2_c	$(x^3yxy)^7$	$2^{11}\cdot 3\cdot 7$	2_b	2_b
2_d	$(xyxy^2)^2$	$2^{10}\cdot 3$	2_c	2_b
3_a	x^8	$2^6\cdot 3^2$	3_b	3_a
3_b	y	$2^3\cdot 3^2\cdot 5$	3_a	3_b
4_a	x^6	$2^{11}\cdot 3\cdot 7$	4_a	4_a
4_b	$(x^4yxy^2)^3$	$2^{10}\cdot 3$	4_c	4_a
4_c	$(x^3y)^3$	$2^9\cdot 3$	4_b	4_b
4_d	$(x^3yxy^2)^2$	2^9	4_d	4_b
4_e	$(x^4yxyx^2y^2)^2$	2^9	4_e	4_b
4_f	x^4yx^2y	2^7	4_e	4_c
4_g	$xyxy^2$	2^6	4_f	4_c
5_a	$(x^2y)^3$	$2\cdot 3\cdot 5$	5_a	5_a
6_a	x^4	$2^6\cdot 3^2$	6_b	6_a
6_b	$(x^3y)^2$	$2^3\cdot 3^2\cdot 5$	6_a	6_b
6_c	$(x^4yxy^2)^2$	$2^5\cdot 3$	6_d	6_a
6_d	$x^6y^2x^2y$	$2^4\cdot 3$	6_d	6_c
6_e	x^4y^2xy	$2^3\cdot 3$	6_e	6_c
7_a	xy	$2^3\cdot 7$	7_a	7_a
7_b	$(xy)^3$	$2^3\cdot 7$	7_a	7_b
8_a	x^3	$2^7\cdot 3$	8_a	8_a
8_b	$x^4yx^3y^2$	2^6	8_c	8_a
8_c	x^2yx^2yxyxy	2^5	8_b	8_b
8_d	$x^4yxyx^2y^2$	2^5	8_e	8_b
8_e	x^3yxy^2	2^5	8_d	8_b
8_f	$x^4y^2x^2yxy$	2^4	8_e	8_c
10_a	$(x^2y^2)^3$	$2\cdot 3\cdot 5$	10_a	10_a
12_a	$(x)^2$	$2^5\cdot 3$	12_a	12_b
12_b	$(x)^{10}$	$2^5\cdot 3$	12_b	12_a
12_c	x^4yxy^2	$2^4\cdot 3$	12_e	12_a
12_d	$(x^4yxy^2)^5$	$2^4\cdot 3$	12_d	12_b
12_e	x^3y	$2^2\cdot 3$	12_f	12_c
14_a	$(x^5y)^2$	$2^3\cdot 7$	14_a	14_a
14_b	$(x^5y)^6$	$2^3\cdot 7$	14_a	14_b
14_c	xy^2	$2^2\cdot 7$	14_c	14_b
14_d	x^3yxy	$2^2\cdot 7$	14_c	14_c
14_e	$(x^3yxy)^3$	$2^2\cdot 7$	14_b	14_d
14_f	$(xy^2)^3$	$2^2\cdot 7$	14_b	14_a
15_a	$(x^2y)^7$	$2\cdot 3\cdot 5$	15_a	15_a
15_b	x^2y	$2\cdot 3\cdot 5$	15_b	15_b
24_a	x	$2^3\cdot 3$	24_a	24_b
24_b	x^5	$2^3\cdot 3$	24_b	24_a
28_a	x^5y	$2^2\cdot 7$	28_a	28_a
28_b	$(x^5y)^3$	$2^2\cdot 7$	28_a	28_b
30_a	x^2y^2	$2\cdot 3\cdot 5$	30_a	30_b
30_b	$(x^2y^2)^7$	$2\cdot 3\cdot 5$	30_b	30_a

Table 9.4

Conjugacy classes of $H_0 = \langle k_1, k_2, k_3, k_4, k_5, d_1, d_2 \rangle$

i	m_i	$\lvert C_{H_0}(m_i) \rvert$
1_a	1	$2^6 \cdot 3^2$
2_a	$(k_1)^2$	$2^6 \cdot 3^2$
2_b	$k_1 k_3$	2^5
2_c	k_5	2^3
3_a	d_1	$2^3 \cdot 3^2$
3_b	d_2	$2^3 \cdot 3^2$
3_c	$d_1 d_2$	$2 \cdot 3^2$
3_d	$d_1^2 d_2$	$2 \cdot 3^2$
4_a	k_1	$2^5 \cdot 3$
4_b	k_3	$2^5 \cdot 3$
4_c	$k_1 k_4 k_5$	2^4
4_d	$k_2 k_4 k_5$	2^4
6_a	$(k_1 d_2)^2$	$2^3 \cdot 3^2$
6_b	$(k_3 d_1)^2$	$2^3 \cdot 3^2$
6_c	$k_1^2 d_1 d_2$	$2 \cdot 3^2$
6_d	$k_1 d_1^2 d_2$	$2 \cdot 3^2$
8_a	$k_1 k_5$	2^3
8_b	$k_4 k_5$	2^3
12_a	$k_1 d_2$	$2^2 \cdot 3$
12_b	$k_3 d_1$	$2^2 \cdot 3$

Table 9.5

Conjugacy classes of $E_0 = \langle k_1, k_2, k_3, k_4, k_5, d_5, t_0 \rangle$

i	m_i	$\lvert C_{E_0}(m_i) \rvert$
1_a	1	$2^6 \cdot 3 \cdot 7$
2_a	$(k_1)^2$	$2^6 \cdot 3$
2_b	k_5	2^4
3_a	d_5	$2 \cdot 3$
4_a	k_1	2^5
4_b	k_3	2^5
6_a	$k_1 t_0$	$2 \cdot 3$
7_a	t_0	7
7_b	$(t_0)^3$	7
8_a	$k_1 k_5$	2^3
8_b	$k_2 t_0$	2^3

Table 9.6

Conjugacy classes of $N_0 = N_G(d_0) = \langle k_1, k_2, k_3, k_4, k_5, d_1, d_2, s_0, k_6, d_0 \rangle$

i	m_i	$\lvert C_{N_G(d_0)}(m_i) \rvert$
1_a	1	$2^7 \cdot 3^7 \cdot 7 \cdot 13$
2_a	$(k_1)^2$	$2^7 \cdot 3^3$
2_b	k_6	$2^4 \cdot 3^3 \cdot 7$
3_a	d_0	$2^6 \cdot 3^7 \cdot 7 \cdot 13$
3_b	d_1	$2^3 \cdot 3^7$
3_c	$d_1 d_0$	$2^3 \cdot 3^7$
3_d	$d_2 d_0$	$2^3 \cdot 3^7$
3_e	$(k_1 d_2 s_0 k_6)^6$	$2 \cdot 3^7$
3_f	$k_2 d_2 s_0 d_1^2 s_0 d_2 s_0 d_0$	3^7
3_g	$k_2 s_0$	$2^2 \cdot 3^5$
3_h	$d_1 d_2$	$2^2 \cdot 3^5$
3_i	$k_2 s_0 d_0$	$2 \cdot 3^5$
3_j	$d_1 d_2 d_0$	$2 \cdot 3^5$
4_a	k_1	$2^5 \cdot 3^2$
4_b	$k_1 k_6$	$2^4 \cdot 3$
6_a	$(k_1 d_0)^2$	$2^6 \cdot 3^3$
6_b	$(k_1 d_2)^2$	$2^3 \cdot 3^3$
6_c	$(k_1 d_2 d_0)^2$	$2^3 \cdot 3^3$
6_d	$(k_3 d_1 d_0)^2$	$2^3 \cdot 3^3$
6_e	$d_2 s_0$	$2^2 \cdot 3^3$
6_f	$(k_5 d_1 k_6)^2$	$2^2 \cdot 3^3$
6_g	$d_2 s_0 d_0$	$2 \cdot 3^3$
6_h	$(k_1 d_2 s_0 k_6)^3$	$2 \cdot 3^3$
6_i	$k_1^2 d_1 d_2 d_0$	$2 \cdot 3^3$
6_j	$d_1 k_6$	$2^2 \cdot 3^2$
6_k	$d_2 k_6$	$2^2 \cdot 3^2$
7_a	s_0	$2 \cdot 3 \cdot 7$
8_a	$k_1 k_5$	$2^3 \cdot 3$
9_a	$(k_1 d_2 s_0 k_6)^2$	$2 \cdot 3^4$
9_b	$(k_1 d_2 s_0 k_6)^4$	$2 \cdot 3^4$
9_c	$(k_1 s_0 d_1 k_6)^2$	$2 \cdot 3^4$
9_d	$k_1 d_1 s_0 d_2 s_0 d_0$	3^4
9_e	$k_4 s_0 d_1 s_0 d_2 d_0$	3^4
9_f	$k_1 s_0 k_2 d_1 s_0 d_1 d_0$	3^4
12_a	$k_1 d_0$	$2^5 \cdot 3^2$
12_b	$k_3 d_0$	$2^5 \cdot 3^2$
12_c	$k_1 d_2$	$2^2 \cdot 3^2$
12_d	$k_1 d_2 d_0$	$2^2 \cdot 3^2$
12_e	$k_3 d_1 d_0$	$2^2 \cdot 3^2$
12_f	$k_5 d_1 k_6$	$2^2 \cdot 3$
12_g	$k_5 k_6 d_1$	$2^2 \cdot 3$
13_a	$d_1 s_0$	$3 \cdot 13$
14_a	$s_0 k_6$	$2 \cdot 7$
18_a	$k_1 d_2 s_0 k_6$	$2 \cdot 3^2$
18_b	$k_1 s_0 d_1 k_6$	$2 \cdot 3^2$
18_c	$k_1 s_0 d_2 k_6$	$2 \cdot 3^2$
21_a	$s_0 d_0$	$3 \cdot 7$
24_a	$k_1 k_5 d_0$	$2^3 \cdot 3$
24_b	$k_3 s_0 d_0$	$2^3 \cdot 3$
39_a	$d_1 s_0 d_0$	$3 \cdot 13$
39_d	$k_1 s_0 d_1 d_0$	$3 \cdot 13$

Table 9.7

Conjugacy classes of $N_4 = N_G(d_4) = \langle d_0, d_5, k_9, k_{10}, k_8, d_4, b_1, b_2, b_3, b_4 \rangle$

i	m_i	$\lvert C_{N_G(d_4)}(m_i) \rvert$
1_a	1	$2^5 \cdot 3^7 \cdot 5$
2_a	$(k_9)^4$	$2^5 \cdot 3^3 \cdot 5$
2_b	k_8	$2^4 \cdot 3^3$
3_a	d_4	$2^4 \cdot 3^7 \cdot 5$
3_b	b_1	$2 \cdot 3^7$
3_c	$d_4 b_1$	3^7
3_d	d_0	$2^2 \cdot 3^5$
3_e	$d_0 d_5$	$2^2 \cdot 3^5$
3_f	$d_0 d_4$	$2 \cdot 3^5$
3_g	$d_5 b_1$	$2 \cdot 3^5$
3_h	$(d_5 b_1)^2$	$2 \cdot 3^5$
3_i	$d_0 d_5 d_4$	$2 \cdot 3^5$
3_j	$d_0 b_1$	3^5
3_k	$d_0 d_5 b_1$	3^5
3_l	$(d_0 d_5 b_1)^2$	3^5
3_m	$d_0 d_4 b_1$	3^5
3_n	$d_0 d_4 b_3$	3^5
3_o	$d_5 d_4 b_1$	3^5
3_p	$(d_5 d_4 b_1)^2$	3^5
3_q	$d_0 d_5 d_4 b_1$	3^5
3_r	$(d_0 d_5 d_4 b_1)^2$	3^5
3_s	$d_0 d_5 d_4 b_3$	3^5
3_t	$(d_0 d_5 d_4 b_3)^2$	3^5
4_a	$(k_9)^2$	$2^4 \cdot 3$
4_b	$(d_0 d_5 k_8)^3$	$2^4 \cdot 3$
5_a	$(d_0 k_{10})^2$	$2 \cdot 3 \cdot 5$
6_a	$(k_9 d_4)^4$	$2^4 \cdot 3^3 \cdot 5$
6_b	$d_0 d_5 k_{10}$	$2^2 \cdot 3^3$
6_c	$d_0 k_{10}^2$	$2^2 \cdot 3^3$
6_d	$k_8 b_1$	$2 \cdot 3^3$
6_e	$d_0 d_5 k_{10} d_4$	$2 \cdot 3^3$
6_f	$d_0 k_{10}^2 d_4$	$2 \cdot 3^3$
6_g	$k_9 k_8$	$2^2 \cdot 3^2$
6_h	$(k_9 k_8)^5$	$2^2 \cdot 3^2$
6_i	$k_9 k_8 b_1$	$2 \cdot 3^2$
6_j	$(k_9 k_8 b_1)^5$	$2 \cdot 3^2$
8_a	k_9	$2^3 \cdot 3$
8_b	$d_0 k_8$	2^3
10_a	$d_0 k_{10}$	$2 \cdot 3 \cdot 5$
12_a	$(k_9 d_4)^2$	$2^3 \cdot 3$
12_b	$d_0 d_5 k_8$	$2^2 \cdot 3$
12_c	$(d_0 d_5 k_8)^5$	$2^2 \cdot 3$
15_a	$(d_0 k_{10} d_4)^2$	$2 \cdot 3 \cdot 5$
15_b	$(d_0 k_{10} d_4)^{14}$	$2 \cdot 3 \cdot 5$
24_a	$k_9 d_4$	$2^3 \cdot 3$
24_b	$(k_9 d_4)^{13}$	$2^3 \cdot 3$
30_a	$d_0 k_{10} d_4$	$2 \cdot 3 \cdot 5$
30_b	$(d_0 k_{10} d_4)^7$	$2 \cdot 3 \cdot 5$

Table 9.8

Conjugacy classes of $N_1 = N_G(D_1) = \langle d_{14}, d_{15}, b_5, b_6, k_9, d_3 \rangle$

i	m_i	$\lvert C_{N_G(D_1)}(m_i) \rvert$
1_a	1	$2^4 \cdot 3^{10}$
2_a	$(k_9)^4$	$2^4 \cdot 3^4$
2_b	$k_9 d_3$	$2^2 \cdot 3^4$
3_a	$(d_{14} b_5)^3$	$2 \cdot 3^{10}$
3_b	$(d_{14} k_9)^8$	$2^3 \cdot 3^7$
3_c	$(d_{14} d_3 b_5)^3$	$2 \cdot 3^8$
3_d	$(d_{14} d_3 k_9)^2$	$2^2 \cdot 3^7$
3_e	b_5	$2 \cdot 3^6$
3_f	$(d_{14} k_9 d_3)^2$	$2 \cdot 3^6$
3_g	$(d_{14} b_5 d_3 k_9)^2$	$2 \cdot 3^6$
3_h	d_{14}	$2 \cdot 3^5$
3_i	$d_{14} d_{15} b_5$	3^5
3_j	d_3	$2 \cdot 3^4$
3_k	$d_{14} b_5 d_3$	3^4
4_a	$(k_9)^2$	$2^3 \cdot 3$
6_a	$(d_{14} k_9)^4$	$2^3 \cdot 3^3$
6_b	$(d_{15} k_9 d_3)^3$	$2 \cdot 3^4$
6_c	$(d_{14} k_9^2 d_3^2)^3$	$2 \cdot 3^4$
6_d	$d_{14} d_3 k_9$	$2^2 \cdot 3^3$
6_e	$(d_{14} d_3 k_9)^5$	$2^2 \cdot 3^3$
6_f	$d_{14} k_9^4$	$2^2 \cdot 3^3$
6_g	$d_{14} k_9 d_3$	$2 \cdot 3^3$
6_h	$b_5 d_3 k_9$	$2 \cdot 3^3$
6_i	$d_{14} b_5 d_3 k_9$	$2 \cdot 3^3$
6_j	$d_{15} d_3 k_9$	$2 \cdot 3^2$
6_k	$k_9^2 d_3^2$	$2 \cdot 3^2$
8_a	k_9	$2^3 \cdot 3$
8_b	$(k_9)^5$	$2^3 \cdot 3$
9_a	$(d_{15} k_9 d_3)^2$	$2 \cdot 3^6$
9_b	$(d_{14} d_3)^3$	3^6
9_c	$d_{14} b_5 d_3 d_{15} b_5$	$2 \cdot 3^5$
9_d	$d_{14} b_5$	3^5
9_e	$d_{14}^2 d_3 b_6$	3^5
9_f	$d_{14} d_{15} b_6 d_3$	3^5
9_g	$d_{14} d_3^2 b_5$	3^5
9_h	$d_{15} b_6 d_3 b_6$	3^5
9_i	$d_{14} b_6$	$2 \cdot 3^4$
9_j	$b_5 d_3 b_6$	$2 \cdot 3^4$
9_k	$d_{14} d_3 b_5$	3^4
12_a	$(d_{14} k_9)^2$	$2^3 \cdot 3$
12_b	$(d_{14} k_9)^{10}$	$2^3 \cdot 3$
18_a	$d_{15} k_9 d_3$	$2 \cdot 3^2$
18_b	$b_6 k_9 d_3$	$2 \cdot 3^2$
18_c	$d_{14} k_9^2 d_3^2$	$2 \cdot 3^2$
18_d	$d_{14} d_3 k_9^2 d_3$	$2 \cdot 3^2$
24_a	$d_{14} k_9$	$2^3 \cdot 3$
24_b	$(d_{14} k_9)^5$	$2^3 \cdot 3$
24_c	$(d_{14} k_9)^{13}$	$2^3 \cdot 3$
24_d	$(d_{14} k_9)^{17}$	$2^3 \cdot 3$
27_a	$d_{14} d_3$	3^3
27_b	$(d_{14} d_3)^2$	3^3
27_c	$b_5 d_3$	3^3

Table 9.9
Conjugacy classes of $N_{10} = N_G(d_{10}) = \langle l_1, l_2, l_3, l_4, k_{12}, d_{16}\rangle$

i	m_i	$\lvert C_{N_G(d_{10})}(m_i)\rvert$
1_a	1	$2^4 \cdot 3^{10}$
2_a	$(k_{12})^4$	$2^4 \cdot 3^4$
2_b	$k_{12}d_{16}$	$2^2 \cdot 3^4$
3_a	$(l_1d_{16})^3$	$2^3 \cdot 3^{10}$
3_b	$(l_1)^3$	$2 \cdot 3^9$
3_c	$(l_1k_{12})^8$	$2^4 \cdot 3^7$
3_d	$(l_1k_{12})^{16}$	$2^4 \cdot 3^7$
3_e	$l_1^2l_2$	$2 \cdot 3^7$
3_f	$l_1l_2l_3$	$2 \cdot 3^7$
3_g	$l_2^2l_3$	$2 \cdot 3^7$
3_h	d_{16}	$2 \cdot 3^5$
3_i	l_2l_3	$2 \cdot 3^5$
3_j	$l_1^2d_{16}$	$2 \cdot 3^5$
3_k	$l_1l_2d_{16}$	$2 \cdot 3^5$
3_l	$(l_1k_{12}^2d_{16}^2)^2$	$2 \cdot 3^5$
3_m	l_4	3^5
3_n	l_3d_{16}	3^5
3_o	$(l_3d_{16})^2$	3^5
3_p	$l_1d_{16}l_3d_{16}$	3^5
3_q	$l_1l_4d_{16}l_4^2$	3^5
4_a	$(k_{12})^2$	$2^3 \cdot 3^3$
6_a	$(l_4k_{12}^2)^2$	$2^3 \cdot 3^4$
6_b	$(l_1k_{12})^4$	$2^4 \cdot 3^3$
6_c	$(l_1k_{12})^{20}$	$2^4 \cdot 3^3$
6_d	$(l_1k_{12}d_{16})^3$	$2 \cdot 3^4$
6_e	$l_1^2k_{12}l_3d_{16}$	$2^2 \cdot 3^3$
6_f	$(l_1^2k_{12}l_3d_{16})^5$	$2^2 \cdot 3^3$
6_g	$l_4k_{12}d_{16}$	$2 \cdot 3^3$
6_h	$k_{12}^2d_{16}^2$	$2 \cdot 3^3$
6_i	$l_1^2l_2k_{12}d_{16}$	$2 \cdot 3^3$
6_j	$l_1^2k_{12}l_4d_{16}$	$2 \cdot 3^3$
6_k	$l_1k_{12}d_{16}^2k_{12}$	$2 \cdot 3^3$
6_l	$l_3k_{12}d_{16}^2k_{12}$	$2 \cdot 3^3$
6_m	$l_1d_{16}k_{12}$	$2 \cdot 3^2$
6_n	$l_1k_{12}^2d_{16}^2$	$2 \cdot 3^2$
8_a	k_{12}	$2^3 \cdot 3$
8_b	$(k_{12})^5$	$2^3 \cdot 3$
9_a	$(l_1^2k_{12}^2)^4$	$2^3 \cdot 3^6$
9_b	$(l_4d_{16})^3$	3^6
9_c	l_3	$2 \cdot 3^5$
9_d	$l_1^2l_4$	3^5
9_e	l_1	$2 \cdot 3^4$
9_f	l_1d_{16}	$2 \cdot 3^4$
9_g	$l_1l_3d_{16}$	3^4
9_h	$l_1l_4d_{16}k_{12}^2$	3^4
12_a	$l_4k_{12}^2$	$2^2 \cdot 3^3$
12_b	$(l_1k_{12})^2$	$2^3 \cdot 3^2$
12_c	$(l_1k_{12})^{10}$	$2^3 \cdot 3^2$
18_a	$(l_1^2k_{12}^2)^2$	$2^3 \cdot 3^2$
18_b	$l_1k_{12}d_{16}$	$2 \cdot 3^2$
18_c	$l_2k_{12}d_{16}$	$2 \cdot 3^2$
18_d	$l_4d_{16}k_{12}^2d_{16}$	$2 \cdot 3^2$
24_a	l_1k_{12}	$2^3 \cdot 3$
24_b	$(l_1k_{12})^5$	$2^3 \cdot 3$
24_c	$(l_1k_{12})^7$	$2^3 \cdot 3$
24_d	$(l_1k_{12})^{11}$	$2^3 \cdot 3$
27_a	l_4d_{16}	3^3
27_b	$(l_4d_{16})^2$	3^3
27_c	$l_1d_{16}k_{12}^2$	3^3
36_a	$l_1^2k_{12}^2$	$2^2 \cdot 3^2$
36_b	$l_1l_3k_{12}^2$	$2^2 \cdot 3^2$
36_c	$(l_1l_3k_{12}^2)^5$	$2^2 \cdot 3^2$

Table 9.10
Conjugacy classes of $N_7 = N_G(s_1) = \langle q_2, q_3, q_4, d_{12}\rangle$

i	m_i	$\lvert C_{N_7}(m_i)\rvert$
1_a	1	$2^4 \cdot 3^2 \cdot 7^2$
2_a	q_2	$2^4 \cdot 3 \cdot 7$
2_b	q_4	$2^2 \cdot 3^2$
3_a	q_3	$2^4 \cdot 3^2 \cdot 7$
3_b	$(q_3)^2$	$2^4 \cdot 3^2 \cdot 7$
3_c	d_{12}	$2 \cdot 3^2 \cdot 7$
3_d	$d_{12}q_3$	$2 \cdot 3^2$
3_e	$(d_{12}q_3)^2$	$2 \cdot 3^2$
4_a	$(d_{12}q_2q_4)^2$	$2^3 \cdot 3 \cdot 7$
6_a	q_2q_3	$2^4 \cdot 3$
6_b	$(q_2q_3)^5$	$2^4 \cdot 3$
6_c	q_3q_4	$2^2 \cdot 3^2$
6_d	$(q_3q_4)^5$	$2^2 \cdot 3^2$
6_e	$d_{12}q_2d_{12}q_2q_4$	$2 \cdot 3^2$
6_f	$d_{12}q_2d_{12}q_2q_3q_4$	$2 \cdot 3^2$
6_g	$(d_{12}q_2d_{12}q_2q_3q_4)^5$	$2 \cdot 3^2$
7_a	s_1	$2^3 \cdot 3 \cdot 7^2$
7_b	$d_{12}q_2$	$3 \cdot 7^2$
7_c	$s_1d_{12}q_2$	7^2
7_d	$s_1{d_{12}}^2q_2$	7^2
8_a	$d_{12}q_2q_4$	$2^3 \cdot 3$
8_b	$(d_{12}q_2q_4)^3$	$2^3 \cdot 3$
12_a	$(d_{12}q_2q_3q_4)^2$	$2^3 \cdot 3$
12_b	$(d_{12}q_2q_3q_4)^{10}$	$2^3 \cdot 3$
14_a	s_1q_2	$2^3 \cdot 7$
21_a	s_1d_{12}	$3 \cdot 7$
21_b	$d_{12}q_2q_3$	$3 \cdot 7$
21_c	$(d_{12}q_2q_3)^2$	$3 \cdot 7$
24_a	$d_{12}q_2q_3q_4$	$2^3 \cdot 3$
24_b	$(d_{12}q_2q_3q_4)^5$	$2^3 \cdot 3$
24_c	$(d_{12}q_2q_3q_4)^{13}$	$2^3 \cdot 3$
24_d	$(d_{12}q_2q_3q_4)^{17}$	$2^3 \cdot 3$
28_a	$s_1{d_{12}}^2q_2d_{12}q_2$	$2^2 \cdot 7$

Table 9.11
Conjugacy classes of $N_5 = N_G(f) = \langle f, f_1, w_1, w_2, w, d_{13}\rangle$

i	n_i	$\lvert C_{N_5}(n_i)\rvert$
1_a	1	$2^5 \cdot 3 \cdot 5^3$
2_a	$(w_1)^2$	$2^5 \cdot 3 \cdot 5$
2_b	$(w_2w)^2$	$2^4 \cdot 5$
3_a	d_{13}	$2^2 \cdot 3 \cdot 5$
4_a	$(w)^2$	$2^5 \cdot 3$
4_b	$(w)^6$	$2^5 \cdot 3$
4_c	w_1	$2^4 \cdot 5$
4_d	w_1w_2w	$2^4 \cdot 5$
4_e	$(w_1w_2w)^3$	$2^4 \cdot 5$
4_f	w_2w	2^4
4_g	$(w_2w)^3$	2^4
5_a	f	$2^3 \cdot 3 \cdot 5^3$
5_b	f_1	$2^2 \cdot 5^2$
6_a	w_1d_{13}	$2^2 \cdot 3 \cdot 5$
8_a	w	2^3
8_b	$(w)^3$	2^3
10_a	$(f_1w_1)^2$	$2^3 \cdot 3 \cdot 5$
10_b	$(f_1w_1w_2w)^2$	$2^2 \cdot 5$
12_a	w^2d_{13}	$2^2 \cdot 3$
12_b	$(w^2d_{13})^7$	$2^2 \cdot 3$
15_a	f_1d_{13}	$2 \cdot 3 \cdot 5$
15_b	$(f_1d_{13})^7$	$2 \cdot 3 \cdot 5$
20_a	f_1w_1	$2^2 \cdot 5$
20_b	$f_1w_1w_2w$	$2^2 \cdot 5$
20_c	$(f_1w_1w_2w)^3$	$2^2 \cdot 5$
30_a	$f_1w_1d_{13}$	$2 \cdot 3 \cdot 5$
30_b	$(f_1w_1d_{13})^7$	$2 \cdot 3 \cdot 5$

10. Character tables

Table 10.1
Character table of $H = C_G(z) \cong 2^{1+8}.A_9$

2	15	15	14	10	4	6	3	11	9	10	9	9	6	3	4	6	3	5	3	3	7	5	6	5
3	4	4	1	1	3	3	4	3	1	1	.	.	.	1	3	3	4	1	1	.	1	1	.	.
5	1	1	.	.	1	.	.	.	1	.	.	.	.	1	1	.	.	.	.	.	.	.	.	.
7	1	1	1	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.
	1a	2a	2b	2c	3a	3b	3c	4a	4b	4c	4d	4e	4f	5a	6a	6b	6c	6d	6e	7a	8a	8b	8c	8d
2P	1a	1a	1a	1a	3a	3b	3c	2a	2a	2b	2b	2b	2c	5a	3a	3b	3c	3b	3b	7a	4a	4b	4c	4d
3P	1a	2a	2b	2c	1a	1a	1a	4a	4b	4c	4d	4e	4f	5a	2a	2a	2a	2b	2c	7a	8a	8b	8c	8d
5P	1a	2a	2b	2c	3a	3b	3c	4a	4b	4c	4d	4e	4f	1a	6a	6b	6c	6d	6e	7a	8a	8b	8c	8d
7P	1a	2a	2b	2c	3a	3b	3c	4a	4b	4c	4d	4e	4f	5a	6a	6b	6c	6d	6e	1a	8a	8b	8c	8d
11P	1a	2a	2b	2c	3a	3b	3c	4a	4b	4c	4d	4e	4f	5a	6a	6b	6c	6d	6e	7a	8a	8b	8c	8d
13P	1a	2a	2b	2c	3a	3b	3c	4a	4b	4c	4d	4e	4f	5a	6a	6b	6c	6d	6e	7a	8a	8b	8c	8d
17P	1a	2a	2b	2c	3a	3b	3c	4a	4b	4c	4d	4e	4f	5a	6a	6b	6c	6d	6e	7a	8a	8b	8c	8d
19P	1a	2a	2b	2c	3a	3b	3c	4a	4b	4c	4d	4e	4f	5a	6a	6b	6c	6d	6e	7a	8a	8b	8c	8d
23P	1a	2a	2b	2c	3a	3b	3c	4a	4b	4c	4d	4e	4f	5a	6a	6b	6c	6d	6e	7a	8a	8b	8c	8d
29P	1a	2a	2b	2c	3a	3b	3c	4a	4b	4c	4d	4e	4f	5a	6a	6b	6c	6d	6e	7a	8a	8b	8c	8d
31P	1a	2a	2b	2c	3a	3b	3c	4a	4b	4c	4d	4e	4f	5a	6a	6b	6c	6d	6e	7a	8a	8b	8c	8d
X.1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.2	8	8	8	.	5	2	−1	8	4	.	.	4	.	3	5	2	−1	2	.	1	.	2	.	.
X.3	21	21	21	−3	−3	.	3	21	1	−3	−3	1	1	1	−3	.	3	.	.	.	−3	−1	1	1
X.4	21	21	21	−3	−3	.	3	21	1	−3	−3	1	1	1	−3	.	3	.	.	.	−3	−1	1	1
X.5	27	27	27	3	9	.	.	27	7	3	3	7	−1	2	9	.	.	.	.	−1	3	1	−1	−1
X.6	28	28	28	−4	10	1	1	28	4	−4	−4	4	.	3	10	1	1	1	−1	.	−4	.	.	.
X.7	35	35	35	3	5	2	−1	35	−5	3	3	−5	−1	.	5	2	−1	2	.	.	3	−1	−1	−1
X.8	35	35	35	3	5	2	−1	35	−5	3	3	−5	−1	.	5	2	−1	2	.	.	3	−1	−1	−1
X.9	42	42	42	2	.	3	−3	42	6	2	2	6	2	−3	.	3	−3	3	−1	.	2	.	2	2
X.10	48	48	48	.	6	.	3	48	8	.	.	8	.	−2	6	.	3	.	.	−1	.	.	.	.
X.11	56	56	56	.	11	2	2	56	−4	.	.	−4	.	1	11	2	2	2	.	.	.	−2	.	.
X.12	84	84	84	4	−6	3	3	84	4	4	4	4	.	−1	−6	3	3	3	1	.	4	.	.	.
X.13	105	105	105	1	15	−3	−3	105	5	1	1	5	1	.	15	−3	−3	−3	1	.	1	−1	1	1
X.14	120	120	−8	−8	.	6	3	8	.	8	.	.	.	.	.	6	3	−2	−2	1	.	.	.	.
X.15	120	120	120	8	.	−3	3	120	.	8	8	.	.	.	.	−3	3	−3	−1	1	8	.	.	.
X.16	120	120	−8	−8	.	−3	3	8	.	8	.	.	.	.	.	−3	3	1	1	1	.	.	.	.
X.17	120	120	−8	−8	.	−3	3	8	.	8	.	.	.	.	.	−3	3	1	1	1	.	.	.	.
X.18	128	−128	.	.	−4	8	2	.	.	.	.	.	.	−2	4	−8	−2	.	.	2	.	.	.	.
X.19	128	−128	.	.	−4	8	2	.	.	.	.	.	.	−2	4	−8	−2	.	.	2	.	.	.	.
X.20	135	135	7	7	.	9	.	−9	15	7	−1	−1	−1	.	.	9	.	1	1	2	−1	3	3	−1
X.21	162	162	162	−6	.	.	.	162	6	−6	−6	6	−2	−3	.	.	.	.	.	1	−6	.	−2	−2
X.22	168	168	168	.	−15	.	−3	168	4	.	.	4	.	3	−15	.	−3	.	.	.	.	−2	.	.
X.23	189	189	189	−3	9	.	.	189	−11	−3	−3	−11	1	−1	9	.	.	.	.	.	−3	1	1	1
X.24	216	216	216	.	−9	.	.	216	−4	.	.	−4	.	1	−9	.	.	.	.	−1	.	2	.	.
X.25	405	405	21	−3	.	.	.	−27	−15	−3	5	1	−3	.	.	.	.	.	.	−1	−3	3	1	1
X.26	405	405	21	−3	.	.	.	−27	−15	−3	5	1	−3	.	.	.	.	.	.	−1	−3	3	1	1
X.27	768	−768	.	.	6	.	−6	.	.	.	.	.	.	−2	−6	.	6	.	.	−2	.	.	.	.
X.28	768	−768	.	.	6	.	−6	.	.	.	.	.	.	−2	−6	.	6	.	.	−2	.	.	.	.
X.29	810	810	42	18	.	.	.	−54	30	18	2	−2	2	.	.	.	.	.	.	−2	−6	.	2	−2
X.30	840	840	−56	8	.	6	−6	56	.	−8	.	.	.	.	.	6	−6	−2	2	.	.	.	.	.
X.31	840	840	−56	8	.	−3	−6	56	.	−8	.	.	.	.	.	−3	−6	1	−1	.	.	.	.	.
X.32	840	840	−56	8	.	−3	−6	56	.	−8	.	.	.	.	.	−3	−6	1	−1	.	.	.	.	.
X.33	896	−896	.	.	−16	8	−4	.	.	.	.	.	.	−4	16	−8	4	.	.	.	.	.	.	.
X.34	945	945	49	1	.	9	.	−63	−15	1	9	1	1	.	.	9	.	1	1	.	−7	−3	−3	1
X.35	945	945	49	−7	.	9	.	−63	45	−7	1	−3	1	.	.	9	.	1	−1	.	1	3	−3	1
X.36	945	945	49	17	.	9	.	−63	−15	17	−7	1	−3	.	.	9	.	1	−1	.	1	−3	1	1
X.37	960	960	−64	.	.	12	−3	64	.	.	.	.	.	.	.	12	−3	−4	.	1	.	.	.	.
X.38	960	960	−64	.	.	−6	−3	64	.	.	.	.	.	.	.	−6	−3	2	.	1	.	.	.	.
X.39	960	960	−64	.	.	−6	−3	64	.	.	.	.	.	.	.	−6	−3	2	.	1	.	.	.	.
X.40	1080	1080	56	8	.	−9	.	−72	.	8	8	.	.	.	.	−9	.	−1	−1	2	−8	.	.	.
X.41	1792	−1792	.	.	4	16	−8	.	.	.	.	.	.	2	−4	−16	8	.	.	.	.	.	.	.
X.42	1890	1890	98	10	.	−9	.	−126	30	10	−6	−2	−2	.	.	−9	.	−1	1	.	2	.	−2	2
X.43	1920	−1920	.	.	.	−12	−6	.	.	.	.	.	.	.	.	12	6	.	.	2	.	.	.	.
X.44	1920	−1920	.	.	.	−12	−6	.	.	.	.	.	.	.	.	12	6	.	.	2	.	.	.	.
X.45	2520	2520	−168	24	.	.	9	168	.	−24	.	.	.	.	.	.	9	.	.	.	.	.	.	.
X.46	2560	−2560	.	.	−20	−8	4	.	.	.	.	.	.	.	20	8	−4	.	.	−2	.	.	.	.
X.47	2688	−2688	.	.	12	.	6	.	.	.	.	.	.	−2	−12	.	−6	.	.	.	.	.	.	.
X.48	2688	−2688	.	.	12	.	6	.	.	.	.	.	.	−2	−12	.	−6	.	.	.	.	.	.	.
X.49	2835	2835	147	−21	.	.	.	−189	15	−21	3	−1	−1	.	.	.	.	.	.	.	3	−3	3	−1
X.50	2835	2835	147	3	.	.	.	−189	−45	3	−5	3	3	.	.	.	.	.	.	.	3	3	−1	−1
X.51	3240	3240	−216	−24	.	.	.	216	.	24	.	.	.	.	.	.	.	.	.	−1	.	.	.	.
X.52	3584	−3584	.	.	−4	8	2	.	.	.	.	.	.	4	4	−8	−2	.	.	.	.	.	.	.

(*continued on next page*)

Table 10.1 (*continued*)

2	5	3	1	3	5	5	2	4	4	3	3	2	2	1	1	3	1	2	3	3	3	3	2
3	.	2	2	1	2	2	3	1	1	1	.	.	.	1	1	2	2	.	1	1	1	1	.
5	.	.	.	1	.	.	.	.	.	.	.	.	.	1	1	.	.	1	.	.	.	.	.
7	.	.	.	.	.	.	.	.	.	.	1	1	1	.	.	.	.	.	.	.	.	.	1
	8e	9a	9b	10a	12a	12b	12c	12d	12e	12f	14a	14b	14c	15a	15b	18a	18b	20a	24a	24b	24c	24d	28a
2P	4e	9a	9b	5a	6b	6b	6c	6d	6d	6a	7a	7a	7a	15a	15b	9a	9b	10a	12a	12b	12f	12f	14a
3P	8e	3c	3c	10a	4a	4a	4a	4c	4c	4b	14a	14c	14b	5a	5a	6c	6c	20a	8a	8a	8b	8b	28a
5P	8e	9a	9b	2a	12b	12a	12c	12e	12d	12f	14a	14c	14b	3a	3a	18a	18b	4b	24b	24a	24c	24d	28a
7P	8e	9a	9b	10a	12a	12b	12c	12d	12e	12f	2a	2b	2b	15b	15a	18a	18b	20a	24a	24b	24c	24d	4a
11P	8e	9a	9b	10a	12b	12a	12c	12e	12d	12f	14a	14b	14c	15b	15a	18a	18b	20a	24b	24a	24c	24d	28a
13P	8e	9a	9b	10a	12a	12b	12c	12d	12e	12f	14a	14c	14b	15b	15a	18a	18b	20a	24a	24b	24d	24c	28a
17P	8e	9a	9b	10a	12b	12a	12c	12e	12d	12f	14a	14c	14b	15a	15b	18a	18b	20a	24b	24a	24d	24c	28a
19P	8e	9a	9b	10a	12a	12b	12c	12d	12e	12f	14a	14c	14b	15a	15b	18a	18b	20a	24a	24b	24d	24c	28a
23P	8e	9a	9b	10a	12b	12a	12c	12e	12d	12f	14a	14b	14c	15a	15b	18a	18b	20a	24b	24a	24d	24c	28a
29P	8e	9a	9b	10a	12b	12a	12c	12e	12d	12f	14a	14b	14c	15b	15a	18a	18b	20a	24b	24a	24c	24d	28a
31P	8e	9a	9b	10a	12a	12b	12c	12d	12e	12f	14a	14c	14b	15a	15b	18a	18b	20a	24a	24b	24c	24d	28a
X.1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.2	2	−1	−1	3	2	2	−1	.	.	1	1	1	1	.	.	−1	−1	−1	.	.	−1	−1	1
X.3	−1	.	.	1	.	.	3	.	.	1	.	.	.	E	/E	.	.	1	.	.	−1	−1	.
X.4	−1	.	.	1	.	.	3	.	.	1	.	.	.	/E	E	.	.	1	.	.	−1	−1	.
X.5	1	.	.	2	.	.	.	.	.	1	−1	−1	−1	−1	−1	.	.	2	.	.	1	1	−1
X.6	.	1	1	3	1	1	1	−1	−1	−2	.	.	.	.	.	1	1	−1	−1	−1	.	.	.
X.7	−1	−1	2	.	2	2	−1	.	.	1	.	.	.	.	.	−1	2	.	.	.	−1	−1	.
X.8	−1	2	−1	.	2	2	−1	.	.	1	.	.	.	.	.	2	−1	.	.	.	−1	−1	.
X.9	.	.	.	−3	3	3	−3	−1	−1	.	.	.	.	.	.	.	.	1	−1	−1	.	.	.
X.10	.	.	.	−2	.	.	3	.	.	2	−1	−1	−1	1	1	.	.	−2	.	.	.	.	−1
X.11	−2	−1	−1	1	2	2	2	.	.	−1	.	.	.	1	1	−1	−1	1	.	.	1	1	.
X.12	.	.	.	−1	3	3	3	1	1	−2	.	.	.	−1	−1	.	.	−1	1	1	.	.	.
X.13	−1	.	.	.	−3	−3	−3	1	1	−1	.	.	.	.	.	.	.	.	1	1	−1	−1	.
X.14	.	3	.	.	2	2	−1	2	2	.	1	−1	−1	.	.	3	.	.	.	.	.	.	1
X.15	.	.	.	.	−3	−3	3	−1	−1	.	1	1	1	.	.	.	.	.	−1	−1	.	.	1
X.16	.	.	.	.	A	/A	−1	−1	−1	.	1	−1	−1	.	.	.	.	.	F	−F	.	.	1
X.17	.	.	.	.	/A	A	−1	−1	−1	.	1	−1	−1	.	.	.	.	.	−F	F	.	.	1
X.18	.	2	2	2	.	.	.	.	.	.	−2	.	.	1	1	−2	−2	.	.	.	.	.	.
X.19	.	−4	−1	2	.	.	.	.	.	.	−2	.	.	1	1	4	1	.	.	.	.	.	.
X.20	−1	.	.	.	−3	−3	.	1	1	.	2	.	.	.	.	.	.	.	−1	−1	.	.	−2
X.21	.	.	.	−3	.	.	.	.	.	.	1	1	1	.	.	.	.	1	.	.	.	.	1
X.22	−2	.	.	3	.	.	−3	.	.	1	.	.	.	.	.	.	.	−1	.	.	1	1	.
X.23	1	.	.	−1	.	.	.	.	.	1	.	.	.	−1	−1	.	.	−1	.	.	1	1	.
X.24	2	.	.	1	.	.	.	.	.	−1	−1	−1	−1	1	1	.	.	1	.	.	−1	−1	−1
X.25	−1	.	.	.	.	.	.	.	.	.	−1	D	−D	.	.	.	.	.	.	.	.	.	1
X.26	−1	.	.	.	.	.	.	.	.	.	−1	−D	D	.	.	.	.	.	.	.	.	.	1
X.27	.	.	.	2	.	.	.	.	.	.	2	.	.	1	1	.	.	.	.	.	G	−G	.
X.28	.	.	.	2	.	.	.	.	.	.	2	.	.	1	1	.	.	.	.	.	−G	G	.
X.29	.	.	.	.	.	.	.	.	.	.	−2	.	.	.	.	.	.	.	.	.	.	.	2
X.30	.	3	.	.	2	2	2	−2	−2	.	.	.	.	.	.	3	.	.	.	.	.	.	.
X.31	.	.	.	.	A	/A	2	1	1	.	.	.	.	.	.	.	.	.	−F	F	.	.	.
X.32	.	.	.	.	/A	A	2	1	1	.	.	.	.	.	.	.	.	.	F	−F	.	.	.
X.33	.	2	−1	4	.	.	.	.	.	.	.	.	.	−1	−1	−2	1	.	.	.	.	.	.
X.34	1	.	.	.	−3	−3	.	1	1	.	.	.	.	.	.	.	.	.	−1	−1	.	.	.
X.35	−1	.	.	.	−3	−3	.	−1	−1	.	.	.	.	.	.	.	.	.	1	1	.	.	.
X.36	1	.	.	.	−3	−3	.	−1	−1	.	.	.	.	.	.	.	.	.	1	1	.	.	.
X.37	.	−3	.	.	4	4	1	.	.	.	1	−1	−1	.	.	−3	.	.	.	.	.	.	1
X.38	.	.	.	.	B	/B	1	.	.	.	1	−1	−1	.	.	.	.	.	.	.	.	.	1
X.39	.	.	.	.	/B	B	1	.	.	.	1	−1	−1	.	.	.	.	.	.	.	.	.	1
X.40	.	.	.	.	3	3	.	−1	−1	.	2	.	.	.	.	.	.	.	1	1	.	.	−2
X.41	.	−2	1	−2	.	.	.	.	.	.	.	.	.	−1	−1	2	−1	.	.	.	.	.	.
X.42	.	.	.	.	3	3	.	1	1	.	.	.	.	.	.	.	.	.	−1	−1	.	.	.
X.43	.	.	.	.	.	.	.	C	−C	.	−2	.	.	.	.	.	.	.	.	.	.	.	.
X.44	.	.	.	.	.	.	.	−C	C	.	−2	.	.	.	.	.	.	.	.	.	.	.	.
X.45	.	.	.	.	.	.	−3	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
X.46	.	−2	1	.	.	.	.	.	.	.	2	.	.	.	.	2	−1	.	.	.	.	.	.
X.47	.	.	.	2	.	.	.	.	.	.	.	.	.	E	/E	.	.	.	.	.	.	.	.
X.48	.	.	.	2	.	.	.	.	.	.	.	.	.	/E	E	.	.	.	.	.	.	.	.
X.49	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
X.50	−1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
X.51	.	.	.	.	.	.	.	.	.	.	−1	1	1	.	.	.	.	.	.	.	.	.	−1
X.52	.	2	−1	−4	.	.	.	.	.	.	.	.	.	1	1	−2	1	.	.	.	.	.	.

(*continued on next page*)

Table 10.1 (*continued*)

2	1	1	2	2	2
3	1	1	2	2	2
5	1	1	.	.	.
7	.	.	.	.	.
	$30a$	$30b$	$36a$	$36b$	$36c$
$2P$	$15a$	$15b$	$18a$	$18a$	$18a$
$3P$	$10a$	$10a$	$12c$	$12c$	$12c$
$5P$	$6a$	$6a$	$36c$	$36b$	$36a$
$7P$	$30b$	$30a$	$36a$	$36b$	$36c$
$11P$	$30b$	$30a$	$36c$	$36b$	$36a$
$13P$	$30b$	$30a$	$36a$	$36b$	$36c$
$17P$	$30a$	$30b$	$36c$	$36b$	$36a$
$19P$	$30a$	$30b$	$36a$	$36b$	$36c$
$23P$	$30a$	$30b$	$36c$	$36b$	$36a$
$29P$	$30b$	$30a$	$36c$	$36b$	$36a$
$31P$	$30a$	$30b$	$36a$	$36b$	$36c$
$X.1$	1	1	1	1	1
$X.2$	.	.	-1	-1	-1
$X.3$	E	$/E$	.	.	.
$X.4$	$/E$	E	.	.	.
$X.5$	-1	-1	.	.	.
$X.6$	.	.	1	1	1
$X.7$	.	.	-1	-1	-1
$X.8$	.	.	2	2	2
$X.9$	.	.	.	.	.
$X.10$	1	1	.	.	.
$X.11$	1	1	-1	-1	-1
$X.12$	-1	-1	.	.	.
$X.13$	.	.	.	.	.
$X.14$	.	.	-1	-1	-1
$X.15$	.	.	.	.	.
$X.16$	.	.	H	2	$/H$
$X.17$	.	.	$/H$	2	H
$X.18$	-1	-1	.	.	.
$X.19$	-1	-1	.	.	.
$X.20$	.	.	.	.	.
$X.21$	.	.	.	.	.
$X.22$	.	.	.	.	.
$X.23$	-1	-1	.	.	.
$X.24$	1	1	.	.	.
$X.25$	.	.	.	.	.
$X.26$	.	.	.	.	.
$X.27$	-1	-1	.	.	.
$X.28$	-1	-1	.	.	.
$X.29$	.	.	.	.	.
$X.30$	.	.	-1	-1	-1
$X.31$	.	.	H	2	$/H$
$X.32$	.	.	$/H$	2	H
$X.33$	1	1	.	.	.
$X.34$	.	.	.	.	.
$X.35$	.	.	.	.	.
$X.36$	.	.	.	.	.
$X.37$	.	.	1	1	1
$X.38$	.	.	$-H$	-2	$-/H$
$X.39$	.	.	$-/H$	-2	$-H$
$X.40$	.	.	.	.	.
$X.41$	1	1	.	.	.
$X.42$	.	.	.	.	.
$X.43$	.	.	.	.	.
$X.44$	.	.	.	.	.
$X.45$	.	.	.	.	.
$X.46$	.	.	.	.	.
$X.47$	$-E$	$-/E$	.	.	.
$X.48$	$-/E$	$-E$	.	.	.
$X.49$	.	.	.	.	.
$X.50$	.	.	.	.	.
$X.51$	.	.	.	.	.
$X.52$	-1	-1	.	.	.

Notes. Here $A = -1 - 2i\sqrt{3}$, $B = -2 - 4i\sqrt{3}$, $C = -2i\sqrt{3}$, $D = -i\sqrt{7}$, $E = -\frac{1}{2}(1 + i\sqrt{15})$, $F = -i\sqrt{3}$, $G = -i\sqrt{6}$ and $H = -1 + i\sqrt{3}$.

Table 10.2
Character table of $E = N_G(A) \cong 2^5.\mathrm{GL}_5(2)$

2	15	15	11	6	3	11	9	7	1	6	3	4	3	3	7	5	4	1	5	5	2	3	3	2
3	2	2	1	2	2	1	1	.	1	2	2	1	1	1	1	.	.	1	1	1	1	.	.	.
5	1	1	.	.	1	.	.	.	1	.	1	.	.	.	.	.	.	1	.	.	.	.	.	.
7	1	1	1	1	.	1	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	1	1	1
31	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
	$1a$	$2a$	$2b$	$3a$	$3b$	$4a$	$4b$	$4c$	$5a$	$6a$	$6b$	$6c$	$7a$	$7b$	$8a$	$8b$	$8c$	$10a$	$12a$	$12b$	$12c$	$14a$	$14b$	$14c$
$2P$	$1a$	$1a$	$1a$	$3a$	$3b$	$2a$	$2a$	$2b$	$5a$	$3a$	$3b$	$3a$	$7a$	$7b$	$4a$	$4b$	$4c$	$5a$	$6a$	$6a$	$6b$	$7a$	$7b$	$7a$
$3P$	$1a$	$2a$	$2b$	$1a$	$1a$	$4a$	$4b$	$4c$	$5a$	$2a$	$2a$	$2b$	$7b$	$7a$	$8a$	$8b$	$8c$	$10a$	$4a$	$4a$	$4b$	$14b$	$14a$	$14d$
$5P$	$1a$	$2a$	$2b$	$3a$	$3b$	$4a$	$4b$	$4c$	$1a$	$6a$	$6b$	$6c$	$7b$	$7a$	$8a$	$8b$	$8c$	$2a$	$12b$	$12a$	$12c$	$14b$	$14a$	$14d$
$7P$	$1a$	$2a$	$2b$	$3a$	$3b$	$4a$	$4b$	$4c$	$5a$	$6a$	$6b$	$6c$	$1a$	$1a$	$8a$	$8b$	$8c$	$10a$	$12a$	$12b$	$12c$	$2a$	$2a$	$2b$
$11P$	$1a$	$2a$	$2b$	$3a$	$3b$	$4a$	$4b$	$4c$	$5a$	$6a$	$6b$	$6c$	$7a$	$7b$	$8a$	$8b$	$8c$	$10a$	$12b$	$12a$	$12c$	$14a$	$14b$	$14c$
$13P$	$1a$	$2a$	$2b$	$3a$	$3b$	$4a$	$4b$	$4c$	$5a$	$6a$	$6b$	$6c$	$7b$	$7a$	$8a$	$8b$	$8c$	$10a$	$12a$	$12b$	$12c$	$14b$	$14a$	$14d$
$17P$	$1a$	$2a$	$2b$	$3a$	$3b$	$4a$	$4b$	$4c$	$5a$	$6a$	$6b$	$6c$	$7b$	$7a$	$8a$	$8b$	$8c$	$10a$	$12b$	$12a$	$12c$	$14b$	$14a$	$14d$
$19P$	$1a$	$2a$	$2b$	$3a$	$3b$	$4a$	$4b$	$4c$	$5a$	$6a$	$6b$	$6c$	$7b$	$7a$	$8a$	$8b$	$8c$	$10a$	$12a$	$12b$	$12c$	$14b$	$14a$	$14d$
$23P$	$1a$	$2a$	$2b$	$3a$	$3b$	$4a$	$4b$	$4c$	$5a$	$6a$	$6b$	$6c$	$7a$	$7b$	$8a$	$8b$	$8c$	$10a$	$12b$	$12a$	$12c$	$14a$	$14b$	$14c$
$29P$	$1a$	$2a$	$2b$	$3a$	$3b$	$4a$	$4b$	$4c$	$5a$	$6a$	$6b$	$6c$	$7a$	$7b$	$8a$	$8b$	$8c$	$10a$	$12b$	$12a$	$12c$	$14a$	$14b$	$14c$
$31P$	$1a$	$2a$	$2b$	$3a$	$3b$	$4a$	$4b$	$4c$	$5a$	$6a$	$6b$	$6c$	$7b$	$7a$	$8a$	$8b$	$8c$	$10a$	$12a$	$12b$	$12c$	$14b$	$14a$	$14d$
$X.1$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$X.2$	30	30	14	6	.	14	6	2	.	6	.	2	2	2	6	2	.	.	2	2	.	2	2	.
$X.3$	124	124	28	1	4	28	12	4	−1	1	4	1	−2	−2	4	.	.	−1	1	1	.	−2	−2	.
$X.4$	155	155	27	8	5	27	−5	−5	.	8	5	.	1	1	3	−1	−1	.	.	.	1	1	1	−1
$X.5$	217	217	−7	7	4	−7	9	1	2	7	4	−1	.	.	−7	1	1	2	−1	−1	.	.	.	.
$X.6$	248	−8	−8	14	−4	8	.	.	−2	−2	4	−2	3	3	.	.	.	2	2	2	.	−1	−1	−1
$X.7$	280	280	56	7	−5	56	8	.	.	7	−5	−1	.	.	8	.	.	.	−1	−1	−1	.	.	.
$X.8$	315	315	−21	.	.	−21	3	−1	.	.	.	.	.	.	3	−1	1	.	.	.	.	.	.	.
$X.9$	315	315	−21	.	.	−21	3	−1	.	.	.	.	.	.	3	−1	1	.	.	.	.	.	.	.
$X.10$	315	315	−21	.	.	−21	3	−1	.	.	.	.	.	.	3	−1	1	.	.	.	.	.	.	.
$X.11$	315	315	−21	.	.	−21	3	−1	.	.	.	.	.	.	3	−1	1	.	.	.	.	.	.	.
$X.12$	315	315	−21	.	.	−21	3	−1	.	.	.	.	.	.	3	−1	1	.	.	.	.	.	.	.
$X.13$	315	315	−21	.	.	−21	3	−1	.	.	.	.	.	.	3	−1	1	.	.	.	.	.	.	.
$X.14$	465	465	−31	3	.	−31	9	−3	.	3	.	−1	A	$/A$	1	1	−1	.	−1	−1	.	A	$/A$	$-A$
$X.15$	465	465	−31	3	.	−31	9	−3	.	3	.	−1	$/A$	A	1	1	−1	.	−1	−1	.	$/A$	A	$-/A$
$X.16$	465	465	17	3	.	17	−15	1	.	3	.	−1	A	$/A$	1	1	1	.	−1	−1	.	A	$/A$	A
$X.17$	465	465	17	3	.	17	−15	1	.	3	.	−1	$/A$	A	1	1	1	.	−1	−1	.	$/A$	A	$/A$
$X.18$	496	496	48	−8	1	48	16	.	1	−8	1	.	−1	−1	.	.	.	1	.	.	1	−1	−1	−1
$X.19$	651	651	−21	.	6	−21	−5	3	1	.	6	.	.	.	3	−1	−1	1	.	.	−2	.	.	.
$X.20$	651	651	−21	.	−3	−21	−5	3	1	.	−3	.	.	.	3	−1	−1	1	.	.	1	.	.	.
$X.21$	651	651	−21	.	−3	−21	−5	3	1	.	−3	.	.	.	3	−1	−1	1	.	.	1	.	.	.
$X.22$	744	−24	−24	.	−6	24	.	.	−1	.	6	.	B	$/B$	.	.	.	1	.	.	.	$-A$	$-/A$	$-A$
$X.23$	744	−24	−24	.	−6	24	.	.	−1	.	6	.	$/B$	B	.	.	.	1	.	.	.	$-/A$	$-A$	$-/A$
$X.24$	868	868	−28	7	1	−28	4	4	−2	7	1	−1	.	.	−4	.	.	−2	−1	−1	1	.	.	.
$X.25$	930	930	50	6	.	50	−6	−2	.	6	.	2	−1	−1	−6	−2	.	.	2	2	.	−1	−1	1
$X.26$	930	930	−14	−3	.	−14	−6	−2	.	−3	.	1	C	$/C$	2	2	.	.	1	1	.	C	$/C$	.
$X.27$	930	930	−14	−3	.	−14	−6	−2	.	−3	.	1	$/C$	C	2	2	.	.	1	1	.	$/C$	C	.
$X.28$	960	960	64	−6	.	64	.	.	.	−6	.	−2	1	1	.	.	.	.	−2	−2	.	1	1	1
$X.29$	1024	1024	.	−8	4	.	.	.	−1	−8	4	.	2	2	.	.	.	−1	.	.	.	2	2	.
$X.30$	1240	1240	−8	1	−5	−8	8	.	.	1	−5	1	1	1	−8	.	.	.	1	1	−1	1	1	−1
$X.31$	1488	−48	−48	.	6	48	.	.	−2	.	−6	.	−3	−3	.	.	.	2	.	.	.	1	1	1
$X.32$	1736	−56	−56	−7	−4	56	.	.	1	1	4	1	.	.	.	.	.	−1	D	$/D$	.	.	.	.
$X.33$	1736	−56	−56	−7	−4	56	.	.	1	1	4	1	.	.	.	.	.	−1	$/D$	D	.	.	.	.
$X.34$	1736	−56	−56	14	2	56	.	.	1	−2	−2	−2	.	.	.	.	.	−1	2	2	.	.	.	.
$X.35$	1736	−56	−56	14	2	56	.	.	1	−2	−2	−2	.	.	.	.	.	−1	2	2	.	.	.	.
$X.36$	1984	−64	−64	−14	4	64	.	.	−1	2	−4	2	3	3	.	.	.	1	−2	−2	.	−1	−1	−1
$X.37$	3720	−120	8	42	.	−8	.	.	.	−6	.	2	3	3	.	.	.	.	−2	−2	.	−1	−1	1
$X.38$	3720	−120	8	−21	.	−8	.	.	.	3	.	−1	3	3	.	.	.	.	$-/D$	$-D$	.	−1	−1	1
$X.39$	3720	−120	8	−21	.	−8	.	.	.	3	.	−1	3	3	.	.	.	.	$-D$	$-/D$	.	−1	−1	1
$X.40$	11160	−360	24	.	.	−24	.	.	.	.	.	.	B	$/B$	.	.	.	.	.	.	.	$-A$	$-/A$	A
$X.41$	11160	−360	24	.	.	−24	.	.	.	.	.	.	$/B$	B	.	.	.	.	.	.	.	$-/A$	$-A$	$/A$

(*continued on next page*)

Table 10.2 (*continued*)

2	2	1	1	.	.	3	3	2	2	1	1	.	.	.	.	.	.
3	.	1	1	1	1	1	1	.	.	1	1	.	.	.	.	.	.
5	.	1	1	.	.	.	.	.	.	1	1	.	.	.	.	.	.
7	1	.	.	1	1	.	.	1	1	.	.	.	.	.	.	.	.
31	.	.	.	.	.	.	.	.	.	.	.	1	1	1	1	1	1
	14d	15a	15b	21a	21b	24a	24b	28a	28b	30a	30b	31a	31b	31c	31d	31e	31f
2P	7b	15a	15b	21a	21b	12a	12b	14a	14b	15b	15a	31a	31b	31c	31d	31e	31f
3P	14c	5a	5a	7b	7a	8a	8a	28b	28a	10a	10a	31b	31f	31e	31c	31a	31d
5P	14c	3b	3b	21b	21a	24b	24a	28b	28a	6b	6b	31f	31d	31a	31e	31b	31c
7P	2b	15b	15a	3a	3a	24a	24b	4a	4a	30b	30a	31c	31e	31f	31b	31d	31a
11P	14d	15b	15a	21a	21b	24b	24a	28a	28b	30b	30a	31e	31a	31d	31f	31c	31b
13P	14c	15b	15a	21b	21a	24a	24b	28b	28a	30b	30a	31e	31a	31d	31f	31c	31b
17P	14c	15a	15b	21b	21a	24b	24a	28b	28a	30a	30b	31b	31f	31e	31c	31a	31d
19P	14c	15a	15b	21b	21a	24a	24b	28b	28a	30a	30b	31c	31e	31f	31b	31d	31a
23P	14d	15a	15b	21a	21b	24b	24a	28a	28b	30a	30b	31d	31c	31b	31a	31f	31e
29P	14d	15b	15a	21a	21b	24b	24a	28a	28b	30b	30a	31d	31c	31b	31a	31f	31e
31P	14c	15a	15b	21b	21a	24a	24b	28b	28a	30a	30b	1a	1a	1a	1a	1a	1a
X.1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.2	.	.	.	−1	−1	.	.	.	.	.	.	−1	−1	−1	−1	−1	−1
X.3	.	−1	−1	1	1	1	1	.	.	−1	−1	.	.	.	.	.	.
X.4	−1	.	.	1	1	.	.	−1	−1	.	.	.	.	.	.	.	.
X.5	.	−1	−1	.	.	−1	−1	.	.	−1	−1	.	.	.	.	.	.
X.6	−1	1	1	.	.	.	.	1	1	−1	−1	.	.	.	.	.	.
X.7	.	.	.	.	.	−1	−1	.	.	.	.	1	1	1	1	1	1
X.8	.	.	.	.	.	.	.	.	.	.	.	G	I	/I	/G	/H	H
X.9	.	.	.	.	.	.	.	.	.	.	.	H	/G	G	/H	I	/I
X.10	.	.	.	.	.	.	.	.	.	.	.	I	H	/H	/I	G	/G
X.11	.	.	.	.	.	.	.	.	.	.	.	/H	G	/G	H	/I	I
X.12	.	.	.	.	.	.	.	.	.	.	.	/I	/H	H	I	/G	G
X.13	.	.	.	.	.	.	.	.	.	.	.	/G	/I	I	G	H	/H
X.14	−/A	.	.	A	/A	1	1	−A	−/A	.	.	.	.	.	.	.	.
X.15	−A	.	.	/A	A	1	1	−/A	−A	.	.	.	.	.	.	.	.
X.16	/A	.	.	A	/A	1	1	A	/A	.	.	.	.	.	.	.	.
X.17	A	.	.	/A	A	1	1	/A	A	.	.	.	.	.	.	.	.
X.18	−1	1	1	−1	−1	.	.	−1	−1	1	1	.	.	.	.	.	.
X.19	.	1	1	.	.	.	.	.	.	1	1	.	.	.	.	.	.
X.20	.	E	/E	.	.	.	.	.	.	/E	E	.	.	.	.	.	.
X.21	.	/E	E	.	.	.	.	.	.	E	/E	.	.	.	.	.	.
X.22	−/A	−1	−1	.	.	.	.	A	/A	1	1	.	.	.	.	.	.
X.23	−A	−1	−1	.	.	.	.	/A	A	1	1	.	.	.	.	.	.
X.24	.	1	1	.	.	−1	−1	.	.	1	1	.	.	.	.	.	.
X.25	1	.	.	−1	−1	.	.	1	1	.	.	.	.	.	.	.	.
X.26	.	.	.	−A	−/A	−1	−1	.	.	.	.	.	.	.	.	.	.
X.27	.	.	.	−/A	−A	−1	−1	.	.	.	.	.	.	.	.	.	.
X.28	1	.	.	1	1	.	.	1	1	.	.	−1	−1	−1	−1	−1	−1
X.29	.	−1	−1	−1	−1	.	.	.	.	−1	−1	1	1	1	1	1	1
X.30	−1	.	.	1	1	1	1	−1	−1	.	.	.	.	.	.	.	.
X.31	1	1	1	.	.	.	.	−1	−1	−1	−1	.	.	.	.	.	.
X.32	.	1	1	.	.	F	−F	.	.	−1	−1	.	.	.	.	.	.
X.33	.	1	1	.	.	−F	F	.	.	−1	−1	.	.	.	.	.	.
X.34	.	/E	E	.	.	.	.	.	.	−E	−/E	.	.	.	.	.	.
X.35	.	E	/E	.	.	.	.	.	.	−/E	−E	.	.	.	.	.	.
X.36	−1	−1	−1	.	.	.	.	1	1	1	1	.	.	.	.	.	.
X.37	1	.	.	.	.	.	.	−1	−1	.	.	.	.	.	.	.	.
X.38	1	.	.	.	.	−F	F	−1	−1	.	.	.	.	.	.	.	.
X.39	1	.	.	.	.	F	−F	−1	−1	.	.	.	.	.	.	.	.
X.40	/A	.	.	.	.	.	.	−A	−/A	.	.	.	.	.	.	.	.
X.41	A	.	.	.	.	.	.	−/A	−A	.	.	.	.	.	.	.	.

Notes. Here $A = -\frac{1}{2}(1 + i\sqrt{7})$, $B = -\frac{3}{2}(1 + i\sqrt{7})$, $C = -1 - i\sqrt{7}$, $D = -1 - 2i\sqrt{3}$, $E = \frac{1}{2}(-1 + i\sqrt{15})$, $F = -i\sqrt{3}$, $G = \xi^{15} + \xi^{23} + \xi^{27} + \xi^{29} + \xi^{30}$, $H = \xi^{11} + \xi^{13} + \xi^{21} + \xi^{22} + \xi^{26}$ and $I = \xi^{7} + \xi^{14} + \xi^{19} + \xi^{25} + \xi^{28}$, where ξ is a primitive 31st root of unity.

Table 10.3

Character table of $D = N_H(A)$

2	15	15	14	11	10	6	3	11	10	9	9	9	7	6	1	6	3	5	4	3	3	3	7	6	5	5
3	2	2	1	1	1	2	2	1	1	1	.	.	.	.	1	2	2	1	1	1	.	.	1	.	.	.
5	1	1	.	.	.	.	1	.	.	.	.	.	.	.	1	.	1	.	.	.	.	.	.	.	.	.
7	1	1	1	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.
	$1a$	$2a$	$2b$	$2c$	$2d$	$3a$	$3b$	$4a$	$4b$	$4c$	$4d$	$4e$	$4f$	$4g$	$5a$	$6a$	$6b$	$6c$	$6d$	$6e$	$7a$	$7b$	$8a$	$8b$	$8c$	$8d$
$2P$	$1a$	$1a$	$1a$	$1a$	$1a$	$3a$	$3b$	$2a$	$2b$	$2a$	$2b$	$2b$	$2c$	$2d$	$5a$	$3a$	$3b$	$3a$	$3a$	$3a$	$7a$	$7b$	$4a$	$4b$	$4c$	$4e$
$3P$	$1a$	$2a$	$2b$	$2c$	$2d$	$1a$	$1a$	$4a$	$4b$	$4c$	$4d$	$4e$	$4f$	$4g$	$5a$	$2a$	$2a$	$2b$	$2c$	$2d$	$7b$	$7a$	$8a$	$8b$	$8c$	$8d$
$5P$	$1a$	$2a$	$2b$	$2c$	$2d$	$3a$	$3b$	$4a$	$4b$	$4c$	$4d$	$4e$	$4f$	$4g$	$1a$	$6a$	$6b$	$6c$	$6d$	$6e$	$7b$	$7a$	$8a$	$8b$	$8c$	$8d$
$7P$	$1a$	$2a$	$2b$	$2c$	$2d$	$3a$	$3b$	$4a$	$4b$	$4c$	$4d$	$4e$	$4f$	$4g$	$5a$	$6a$	$6b$	$6c$	$6d$	$6e$	$1a$	$1a$	$8a$	$8b$	$8c$	$8d$
$11P$	$1a$	$2a$	$2b$	$2c$	$2d$	$3a$	$3b$	$4a$	$4b$	$4c$	$4d$	$4e$	$4f$	$4g$	$5a$	$6a$	$6b$	$6c$	$6d$	$6e$	$7a$	$7b$	$8a$	$8b$	$8c$	$8d$
$13P$	$1a$	$2a$	$2b$	$2c$	$2d$	$3a$	$3b$	$4a$	$4b$	$4c$	$4d$	$4e$	$4f$	$4g$	$5a$	$6a$	$6b$	$6c$	$6d$	$6e$	$7b$	$7a$	$8a$	$8b$	$8c$	$8d$
$17P$	$1a$	$2a$	$2b$	$2c$	$2d$	$3a$	$3b$	$4a$	$4b$	$4c$	$4d$	$4e$	$4f$	$4g$	$5a$	$6a$	$6b$	$6c$	$6d$	$6e$	$7b$	$7a$	$8a$	$8b$	$8c$	$8d$
$19P$	$1a$	$2a$	$2b$	$2c$	$2d$	$3a$	$3b$	$4a$	$4b$	$4c$	$4d$	$4e$	$4f$	$4g$	$5a$	$6a$	$6b$	$6c$	$6d$	$6e$	$7b$	$7a$	$8a$	$8b$	$8c$	$8d$
$23P$	$1a$	$2a$	$2b$	$2c$	$2d$	$3a$	$3b$	$4a$	$4b$	$4c$	$4d$	$4e$	$4f$	$4g$	$5a$	$6a$	$6b$	$6c$	$6d$	$6e$	$7a$	$7b$	$8a$	$8b$	$8c$	$8d$
$29P$	$1a$	$2a$	$2b$	$2c$	$2d$	$3a$	$3b$	$4a$	$4b$	$4c$	$4d$	$4e$	$4f$	$4g$	$5a$	$6a$	$6b$	$6c$	$6d$	$6e$	$7a$	$7b$	$8a$	$8b$	$8c$	$8d$
$X.1$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$X.2$	7	7	7	7	−1	1	4	7	−1	3	−1	3	3	−1	2	1	4	1	1	−1	.	.	−1	−1	1	1
$X.3$	14	14	14	14	6	2	−1	14	6	2	6	2	2	2	−1	2	−1	2	2	.	.	.	6	2	.	.
$X.4$	15	15	15	−1	7	3	.	−1	7	3	−1	3	−1	−1	.	3	.	3	−1	1	1	1	−1	3	1	1
$X.5$	20	20	20	20	4	−1	5	20	4	4	4	4	4	.	.	−1	5	−1	−1	1	−1	−1	4	.	.	.
$X.6$	21	21	21	21	−3	.	6	21	−3	1	−3	1	1	1	1	.	6	.	.	.	.	.	−3	1	−1	−1
$X.7$	21	21	21	21	−3	.	−3	21	−3	1	−3	1	1	1	1	.	−3	.	.	.	.	.	−3	1	−1	−1
$X.8$	21	21	21	21	−3	.	−3	21	−3	1	−3	1	1	1	1	.	−3	.	.	.	.	.	−3	1	−1	−1
$X.9$	28	28	28	28	−4	1	1	28	−4	4	−4	4	4	.	−2	1	1	1	1	−1	.	.	−4	.	.	.
$X.10$	35	35	35	35	3	2	5	35	3	−5	3	−5	−5	−1	.	2	5	2	2	.	.	.	3	−1	−1	−1
$X.11$	45	45	45	45	−3	.	.	45	−3	−3	−3	−3	−3	1	.	.	.	.	.	.	A	$/A$	−3	1	1	1
$X.12$	45	45	45	45	−3	.	.	45	−3	−3	−3	−3	−3	1	.	.	.	.	.	.	$/A$	A	−3	1	1	1
$X.13$	45	45	45	−3	−3	.	.	−3	−3	−3	5	−3	1	−3	.	.	.	.	.	.	A	$/A$	−3	1	1	1
$X.14$	45	45	45	−3	−3	.	.	−3	−3	−3	5	−3	1	−3	.	.	.	.	.	.	$/A$	A	−3	1	1	1
$X.15$	56	56	56	56	8	−1	−4	56	8	.	8	.	.	.	1	−1	−4	−1	−1	−1	.	.	8	.	.	.
$X.16$	64	64	64	64	.	−2	4	64	.	.	.	.	.	.	−1	−2	4	−2	−2	.	1	1	.	.	.	.
$X.17$	70	70	70	70	−2	1	−5	70	−2	2	−2	2	2	−2	.	1	−5	1	1	1	.	.	−2	−2	.	.
$X.18$	90	90	90	−6	18	.	.	−6	18	6	2	6	−2	2	.	.	.	.	.	.	−1	−1	−6	2	.	.
$X.19$	105	105	105	−7	1	3	.	−7	1	−3	9	−3	1	1	.	3	.	3	−1	1	.	.	−7	−3	−1	−1
$X.20$	105	105	105	−7	17	3	.	−7	17	−3	−7	−3	1	−3	.	3	.	3	−1	−1	.	.	1	1	−1	−1
$X.21$	105	105	105	−7	−7	3	.	−7	−7	9	1	9	−3	1	.	3	.	3	−1	−1	.	.	1	−3	1	1
$X.22$	120	120	120	−8	8	−3	.	−8	8	.	8	.	.	.	.	−3	.	−3	1	−1	1	1	−8	.	.	.
$X.23$	120	120	−8	8	.	6	.	−8	.	12	.	−4	.	.	.	6	.	−2	2	.	1	1	.	.	2	−2
$X.24$	120	120	−8	−8	−8	6	.	8	8	.	.	.	.	.	.	6	.	−2	−2	−2	1	1	.	.	.	.
$X.25$	120	120	−8	−8	−8	−3	.	8	8	.	.	.	.	.	.	−3	.	1	1	1	1	1	.	.	.	.
$X.26$	120	120	−8	−8	−8	−3	.	8	8	.	.	.	.	.	.	−3	.	1	1	1	1	1	.	.	.	.
$X.27$	128	−128	.	.	.	8	−4	.	.	.	.	.	.	.	−2	−8	4	.	.	.	2	2	.	.	.	.
$X.28$	210	210	210	−14	10	−3	.	−14	10	6	−6	6	−2	−2	.	−3	.	−3	1	1	.	.	2	−2	.	.
$X.29$	315	315	315	−21	3	.	.	−21	3	−9	−5	−9	3	3	.	.	.	.	.	.	.	.	3	−1	1	1
$X.30$	315	315	315	−21	−21	.	.	−21	−21	3	3	3	−1	−1	.	.	.	.	.	.	.	.	3	3	−1	−1
$X.31$	360	360	−24	24	.	.	.	−24	.	−12	.	4	.	.	.	.	.	.	.	.	$/A$	A	.	.	2	−2
$X.32$	360	360	−24	24	.	.	.	−24	.	−12	.	4	.	.	.	.	.	.	.	.	A	$/A$	.	.	2	−2
$X.33$	360	360	−24	−24	−24	.	.	24	24	.	.	.	.	.	.	.	.	.	.	.	$/A$	A	.	.	.	.
$X.34$	360	360	−24	−24	−24	.	.	24	24	.	.	.	.	.	.	.	.	.	.	.	A	$/A$	.	.	.	.
$X.35$	384	−384	.	.	.	.	−6	.	.	.	.	.	.	.	−1	.	6	.	.	.	B	$/B$	.	.	.	.
$X.36$	384	−384	.	.	.	.	−6	.	.	.	.	.	.	.	−1	.	6	.	.	.	$/B$	B	.	.	.	.
$X.37$	720	720	−48	48	.	.	.	−48	.	24	.	−8	.	.	.	.	.	.	.	.	−1	−1	.	.	.	.
$X.38$	768	−768	.	.	.	.	6	.	.	.	.	.	.	.	−2	.	−6	.	.	.	−2	−2	.	.	.	.
$X.39$	840	840	−56	56	.	6	.	−56	.	−12	.	4	.	.	.	6	.	−2	2	.	.	.	.	.	−2	2
$X.40$	840	840	−56	−56	8	6	.	56	−8	.	.	.	.	.	.	6	.	−2	−2	2	.	.	.	.	.	.
$X.41$	840	840	−56	−56	8	−3	.	56	−8	.	.	.	.	.	.	−3	.	1	1	−1	.	.	.	.	.	.
$X.42$	840	840	−56	−56	8	−3	.	56	−8	.	.	.	.	.	.	−3	.	1	1	−1	.	.	.	.	.	.
$X.43$	896	−896	.	.	.	8	2	.	.	.	.	.	.	.	1	−8	−2	.	.	.	.	.	.	.	.	.
$X.44$	896	−896	.	.	.	8	2	.	.	.	.	.	.	.	1	−8	−2	.	.	.	.	.	.	.	.	.
$X.45$	896	−896	.	.	.	−4	−4	.	.	.	.	.	.	.	1	4	4	.	.	.	.	.	.	.	.	.
$X.46$	896	−896	.	.	.	−4	−4	.	.	.	.	.	.	.	1	4	4	.	.	.	.	.	.	.	.	.
$X.47$	960	960	−64	64	.	−6	.	−64	.	.	.	.	.	.	.	−6	.	2	−2	.	1	1	.	.	.	.
$X.48$	1024	−1024	.	.	.	−8	4	.	.	.	.	.	.	.	−1	8	−4	.	.	.	2	2	.	.	.	.

(*continued on next page*)

Table 10.3 (*continued*)

2	5	4	1	5	5	4	4	2	3	3	2	2	2	2	1	1	3	3	2	2	1	1
3	.	.	1	1	1	1	1	1	.	.	.	.	.	.	1	1	1	1	.	.	1	1
5	.	.	1	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	1	1
7	.	.	.	.	.	.	.	.	1	1	1	1	1	1	.	.	.	.	1	1	.	.
	8e	8f	10a	12a	12b	12c	12d	12e	14a	14b	14c	14d	14e	14f	15a	15b	24a	24b	28a	28b	30a	30b
2P	4d	4f	5a	6a	6a	6c	6c	6b	7a	7b	7b	7a	7b	7a	15a	15b	12a	12b	14a	14b	15a	15b
3P	8e	8f	10a	4a	4a	4b	4b	4c	14b	14a	14f	14e	14d	14c	5a	5a	8a	8a	28b	28a	10a	10a
5P	8e	8f	2a	12b	12a	12d	12c	12e	14b	14a	14f	14e	14d	14c	3b	3b	24b	24a	28b	28a	6b	6b
7P	8e	8f	10a	12a	12b	12c	12d	12e	2a	2a	2b	2c	2c	2b	15b	15a	24a	24b	4a	4a	30b	30a
11P	8e	8f	10a	12b	12a	12d	12c	12e	14a	14b	14c	14d	14e	14f	15b	15a	24b	24a	28a	28b	30b	30a
13P	8e	8f	10a	12a	12b	12c	12d	12e	14b	14a	14f	14e	14d	14c	15b	15a	24a	24b	28b	28a	30b	30a
17P	8e	8f	10a	12b	12a	12d	12c	12e	14b	14a	14f	14e	14d	14c	15a	15b	24b	24a	28b	28a	30a	30b
19P	8e	8f	10a	12a	12b	12c	12d	12e	14b	14a	14f	14e	14d	14c	15a	15b	24a	24b	28b	28a	30a	30b
23P	8e	8f	10a	12b	12a	12d	12c	12e	14a	14b	14c	14d	14e	14f	15a	15b	24b	24a	28a	28b	30a	30b
29P	8e	8f	10a	12b	12a	12d	12c	12e	14a	14b	14c	14d	14e	14f	15b	15a	24b	24a	28a	28b	30b	30a
$X.1$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$X.2$	-1	1	2	1	1	-1	-1	.	.	.	.	.	.	.	-1	-1	-1	-1	.	.	-1	-1
$X.3$	2	.	-1	2	2	.	.	-1	.	.	.	.	.	.	-1	-1	.	.	.	.	-1	-1
$X.4$	-1	-1	.	-1	-1	1	1	.	1	1	1	-1	-1	1	.	.	-1	-1	-1	-1	.	.
$X.5$	.	.	.	-1	-1	1	1	1	-1	-1	-1	-1	-1	-1	.	.	1	1	-1	-1	.	.
$X.6$	1	-1	1	.	.	.	.	-2	.	.	.	.	.	.	1	1	.	.	.	.	1	1
$X.7$	1	-1	1	.	.	.	.	1	.	.	.	.	.	.	E	$/E$	.	.	.	.	E	$/E$
$X.8$	1	-1	1	.	.	.	.	1	.	.	.	.	.	.	$/E$	E	.	.	.	.	$/E$	E
$X.9$	.	.	-2	1	1	-1	-1	1	.	.	.	.	.	.	1	1	-1	-1	.	.	1	1
$X.10$	-1	-1	.	2	2	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.11$	1	1	.	.	.	.	.	.	A	$/A$	$/A$	A	$/A$	A	.	.	.	.	A	$/A$	.	.
$X.12$	1	1	.	.	.	.	.	.	$/A$	A	A	$/A$	A	$/A$	.	.	.	.	$/A$	A	.	.
$X.13$	1	-1	.	.	.	.	.	.	A	$/A$	$/A$	$-A$	$-/A$	A	.	.	.	.	$-A$	$-/A$	.	.
$X.14$	1	-1	.	.	.	.	.	.	$/A$	A	A	$-/A$	$-A$	$/A$	.	.	.	.	$-/A$	$-A$	.	.
$X.15$	.	.	1	-1	-1	-1	-1	.	.	.	.	.	.	.	1	1	-1	-1	.	.	1	1
$X.16$	.	.	-1	-2	-2	.	.	.	1	1	1	1	1	1	-1	-1	.	.	1	1	-1	-1
$X.17$	-2	.	.	1	1	1	1	-1	.	.	.	.	.	.	.	.	1	1	.	.	.	.
$X.18$	-2	.	.	.	.	.	.	.	-1	-1	-1	1	1	-1	.	.	.	.	1	1	.	.
$X.19$	1	1	.	-1	-1	1	1	.	.	.	.	.	.	.	.	.	-1	-1	.	.	.	.
$X.20$	1	1	.	-1	-1	-1	-1	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.
$X.21$	1	-1	.	-1	-1	-1	-1	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.
$X.22$	.	.	.	1	1	-1	-1	.	1	1	1	-1	-1	1	.	.	1	1	-1	-1	.	.
$X.23$	.	.	.	-2	-2	.	.	.	1	1	-1	1	1	-1	.	.	.	.	-1	-1	.	.
$X.24$	.	.	.	2	2	2	2	.	1	1	-1	-1	-1	-1	.	.	.	.	1	1	.	.
$X.25$	.	.	.	C	$/C$	-1	-1	.	1	1	-1	-1	-1	-1	.	.	F	$-F$	1	1	.	.
$X.26$	.	.	.	$/C$	C	-1	-1	.	1	1	-1	-1	-1	-1	.	.	$-F$	F	1	1	.	.
$X.27$	.	.	2	.	.	.	.	.	-2	-2	.	.	.	.	1	1	.	.	.	.	-1	-1
$X.28$	2	.	.	1	1	1	1	.	.	.	.	.	.	.	.	.	-1	-1	.	.	.	.
$X.29$	-1	-1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.30$	-1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.31$	.	.	.	.	.	.	.	.	$/A$	A	$-A$	$/A$	A	$-/A$	.	.	.	.	$-/A$	$-A$	.	.
$X.32$	.	.	.	.	.	.	.	.	A	$/A$	$-/A$	A	$/A$	$-A$	.	.	.	.	$-A$	$-/A$	.	.
$X.33$	.	.	.	.	.	.	.	.	$/A$	A	$-A$	$-/A$	$-A$	$-/A$	.	.	.	.	$/A$	A	.	.
$X.34$	.	.	.	.	.	.	.	.	A	$/A$	$-/A$	$-A$	$-/A$	$-A$	.	.	.	.	A	$/A$	.	.
$X.35$	.	.	1	.	.	.	.	.	$-B$	$-/B$	.	.	.	.	-1	-1	.	.	.	.	1	1
$X.36$	.	.	1	.	.	.	.	.	$-/B$	$-B$	.	.	.	.	-1	-1	.	.	.	.	1	1
$X.37$	.	.	.	.	.	.	.	.	-1	-1	1	-1	-1	1	.	.	.	.	1	1	.	.
$X.38$	.	.	2	.	.	.	.	.	2	2	.	.	.	.	1	1	.	.	.	.	-1	-1
$X.39$	.	.	.	-2	-2	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.40$	.	.	.	2	2	-2	-2	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.41$	.	.	.	$/C$	C	1	1	.	.	.	.	.	.	.	.	.	F	$-F$	.	.	.	.
$X.42$	.	.	.	C	$/C$	1	1	.	.	.	.	.	.	.	.	.	$-F$	F	.	.	.	.
$X.43$	.	.	-1	.	.	.	.	.	.	.	.	.	.	.	E	$/E$	.	.	.	.	$-E$	$-/E$
$X.44$	.	.	-1	.	.	.	.	.	.	.	.	.	.	.	$/E$	E	.	.	.	.	$-/E$	$-E$
$X.45$	.	.	-1	.	.	D	$-D$	.	.	.	.	.	.	.	1	1	.	.	.	.	-1	-1
$X.46$	.	.	-1	.	.	$-D$	D	.	.	.	.	.	.	.	1	1	.	.	.	.	-1	-1
$X.47$	.	.	.	2	2	.	.	.	1	1	-1	1	1	-1	.	.	.	.	-1	-1	.	.
$X.48$	.	.	1	.	.	.	.	.	-2	-2	.	.	.	.	-1	-1	.	.	.	.	1	1

Notes. Here $A = -\frac{1}{2}(1 + i\sqrt{7})$, $B = -1 + i\sqrt{7}$, $C = -1 + 2i\sqrt{3}$, $D = -2i\sqrt{3}$, $E = \frac{1}{2}(-1 + i\sqrt{15})$ and $F = i\sqrt{3}$.

Table 10.4

Character table of H_0

2	6	6	5	3	3	3	1	1	5	5	4	4	3	3	1	1	3	3	2	2
3	2	2	.	.	2	2	2	2	1	1	.	.	2	2	2	2	.	.	1	1
	1a	2a	2b	2c	3a	3b	3c	3d	4a	4b	4c	4d	6a	6b	6c	6d	8a	8b	12a	12b
2P	1a	1a	1a	1a	3a	3b	3c	3d	2a	2a	2b	2b	3b	3a	3c	3d	4a	4b	6a	6b
3P	1a	2a	2b	2c	1a	1a	1a	1a	4a	4b	4c	4d	2a	2a	2a	2a	8a	8b	4a	4b
5P	1a	2a	2b	2c	3a	3b	3c	3d	4a	4b	4c	4d	6a	6b	6c	6d	8a	8b	12a	12b
7P	1a	2a	2b	2c	3a	3b	3c	3d	4a	4b	4c	4d	6a	6b	6c	6d	8a	8b	12a	12b
11P	1a	2a	2b	2c	3a	3b	3c	3d	4a	4b	4c	4d	6a	6b	6c	6d	8a	8b	12a	12b
$X.1$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$X.2$	1	1	1	-1	1	1	1	1	1	1	-1	-1	1	1	1	1	-1	-1	1	1
$X.3$	2	2	2	.	2	-1	-1	-1	2	2	.	.	2	-1	-1	-1	.	.	2	-1
$X.4$	2	2	2	.	-1	-1	-1	2	2	2	.	.	-1	-1	2	-1	.	.	-1	-1
$X.5$	2	2	2	.	-1	-1	2	-1	2	2	.	.	-1	-1	-1	2	.	.	-1	-1
$X.6$	2	2	2	.	-1	2	-1	-1	2	2	.	.	-1	2	-1	-1	.	.	-1	2
$X.7$	3	3	-1	1	.	3	.	.	3	-1	-1	-1	.	3	.	.	1	-1	.	-1
$X.8$	3	3	-1	-1	.	3	.	.	3	-1	1	1	.	3	.	.	-1	1	.	-1
$X.9$	3	3	-1	1	3	.	.	.	-1	3	-1	-1	3	.	.	.	-1	1	-1	.
$X.10$	3	3	-1	-1	3	.	.	.	-1	3	1	1	3	.	.	.	1	-1	-1	.
$X.11$	4	-4	.	.	-2	-2	1	1	.	.	2	-2	2	2	-1	-1	.	.	.	.
$X.12$	4	-4	.	.	-2	-2	1	1	.	.	-2	2	2	2	-1	-1	.	.	.	.
$X.13$	6	6	-2	.	.	-3	.	.	6	-2	.	.	.	-3	.	.	.	.	.	1
$X.14$	6	6	-2	.	-3	.	.	.	-2	6	.	.	-3	.	.	.	.	.	1	.
$X.15$	8	-8	.	.	2	-4	-1	-1	.	.	.	.	-2	4	1	1	.	.	.	.
$X.16$	8	-8	.	.	-4	2	-1	-1	.	.	.	.	4	-2	1	1	.	.	.	.
$X.17$	8	-8	.	.	2	2	-1	2	.	.	.	.	-2	-2	-2	1	.	.	.	.
$X.18$	8	-8	.	.	2	2	2	-1	.	.	.	.	-2	-2	1	-2	.	.	.	.
$X.19$	9	9	1	-1	.	.	.	.	-3	-3	-1	-1	.	.	.	.	1	1	.	.
$X.20$	9	9	1	1	.	.	.	.	-3	-3	1	1	.	.	.	.	-1	-1	.	.

Table 10.5

Character table of E_0

2	6	6	4	1	5	5	1	.	.	3	3
3	1	1	.	1	.	.	1	.	.	.	.
7	1	.	.	.	.	.	.	1	1	.	.
	1a	2a	2b	3a	4a	4b	6a	7a	7b	8a	8b
2P	1a	1a	1a	3a	2a	2a	3a	7a	7b	4a	4b
3P	1a	2a	2b	1a	4a	4b	2a	7b	7a	8a	8b
5P	1a	2a	2b	3a	4a	4b	6a	7b	7a	8a	8b
7P	1a	2a	2b	3a	4a	4b	6a	1a	1a	8a	8b
$X.1$	1	1	1	1	1	1	1	1	1	1	1
$X.2$	3	3	-1	.	-1	-1	.	A	$/A$	1	1
$X.3$	3	3	-1	.	-1	-1	.	$/A$	A	1	1
$X.4$	6	6	2	.	2	2	.	-1	-1	.	.
$X.5$	7	-1	-1	1	-1	3	-1	.	.	-1	1
$X.6$	7	-1	-1	1	3	-1	-1	.	.	1	-1
$X.7$	7	7	-1	1	-1	-1	1	.	.	-1	-1
$X.8$	8	8	.	-1	.	.	-1	1	1	.	.
$X.9$	14	-2	-2	-1	2	2	1	.	.	.	.
$X.10$	21	-3	1	.	-3	1	.	.	.	1	-1
$X.11$	21	-3	1	.	1	-3	.	.	.	-1	1

Note. Here $A = -\frac{1}{2}(1 + i\sqrt{7})$.

Table 10.6
Character table of $N_0 = N_G(d_0)$

2	7	7	4	6	3	3	3	1	.	2	2	1	1	5	4	6	3	3	3	2	2	1	1	1	2	2	1	3
3	7	3	3	7	7	7	7	7	7	5	5	5	5	2	1	3	3	3	3	3	3	3	3	3	2	2	1	1
7	1	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.
13	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
	$1a$	$2a$	$2b$	$3a$	$3b$	$3c$	$3d$	$3e$	$3f$	$3g$	$3h$	$3i$	$3j$	$4a$	$4b$	$6a$	$6b$	$6c$	$6d$	$6e$	$6f$	$6g$	$6h$	$6i$	$6j$	$6k$	$7a$	$8a$
$2P$	$1a$	$1a$	$1a$	$3a$	$3c$	$3b$	$3d$	$3e$	$3f$	$3g$	$3h$	$3i$	$3j$	$2a$	$2a$	$3a$	$3d$	$3c$	$3b$	$3g$	$3h$	$3j$	$3e$	$3i$	$3h$	$3h$	$7a$	$4a$
$3P$	$1a$	$2a$	$2b$	$1a$	$1a$	$1a$	$1a$	$1a$	$1a$	$1a$	$1a$	$1a$	$1a$	$4a$	$4b$	$2a$	$2a$	$2a$	$2a$	$2a$	$2a$	$2a$	$2b$	$2a$	$2b$	$2b$	$7a$	$8a$
$7P$	$1a$	$2a$	$2b$	$3a$	$3b$	$3c$	$3d$	$3e$	$3f$	$3g$	$3h$	$3i$	$3j$	$4a$	$4b$	$6a$	$6b$	$6c$	$6d$	$6e$	$6f$	$6g$	$6h$	$6i$	$6j$	$6k$	$1a$	$8a$
$13P$	$1a$	$2a$	$2b$	$3a$	$3b$	$3c$	$3d$	$3e$	$3f$	$3g$	$3h$	$3i$	$3j$	$4a$	$4b$	$6a$	$6b$	$6c$	$6d$	$6e$	$6f$	$6g$	$6h$	$6i$	$6j$	$6k$	$7a$	$8a$
$X.1$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$X.2$	1	1	−1	1	1	1	1	1	1	1	1	1	1	1	−1	1	1	1	1	1	1	1	−1	1	−1	−1	1	1
$X.3$	2	2	.	−1	−1	−1	2	2	−1	2	2	−1	−1	2	.	−1	2	−1	−1	2	2	−1	.	−1	.	.	2	2
$X.4$	14	−2	.	14	5	5	5	−4	−4	2	−1	2	−1	2	.	−2	1	1	1	−2	1	1	.	−2	I	$-I$	.	.
$X.5$	14	−2	.	14	5	5	5	−4	−4	2	−1	2	−1	2	.	−2	1	1	1	−2	1	1	.	−2	$-I$	I	.	.
$X.6$	28	−4	.	−14	−5	−5	10	−8	4	4	−2	−2	1	4	.	2	2	−1	−1	−4	2	−1	.	2	.	.	.	.
$X.7$	64	.	8	64	−8	−8	−8	1	1	4	−2	4	−2	.	.	.	.	.	.	.	.	.	−1	.	J	$/J$	1	.
$X.8$	64	.	8	64	−8	−8	−8	1	1	4	−2	4	−2	.	.	.	.	.	.	.	.	.	−1	.	$/J$	J	1	.
$X.9$	64	.	−8	64	−8	−8	−8	1	1	4	−2	4	−2	.	.	.	.	.	.	.	.	.	1	.	$-/J$	$-J$	1	.
$X.10$	64	.	−8	64	−8	−8	−8	1	1	4	−2	4	−2	.	.	.	.	.	.	.	.	.	1	.	$-J$	$-/J$	1	.
$X.11$	78	−2	−6	78	−3	−3	−3	−3	−3	−3	6	−3	6	2	2	−2	1	1	1	1	−2	−2	3	1	.	.	1	.
$X.12$	78	−2	6	78	−3	−3	−3	−3	−3	−3	6	−3	6	2	−2	−2	1	1	1	1	−2	−2	−3	1	.	.	1	.
$X.13$	91	−5	−7	91	10	10	10	10	10	1	1	1	1	3	1	−5	−2	−2	−2	1	1	1	2	1	−1	−1	.	−1
$X.14$	91	−5	7	91	10	10	10	10	10	1	1	1	1	3	−1	−5	−2	−2	−2	1	1	1	−2	1	1	1	.	−1
$X.15$	104	8	−8	104	14	14	14	5	5	2	−1	2	−1	.	.	8	2	2	2	2	−1	−1	1	2	1	1	−1	.
$X.16$	104	8	8	104	14	14	14	5	5	2	−1	2	−1	.	.	8	2	2	2	2	−1	−1	−1	2	−1	−1	−1	.
$X.17$	128	.	.	−64	8	8	−16	2	−1	8	−4	−4	2	.	.	.	.	.	.	.	.	.	.	.	.	.	2	.
$X.18$	128	.	.	−64	8	8	−16	2	−1	8	−4	−4	2	.	.	.	.	.	.	.	.	.	.	.	.	.	2	.
$X.19$	156	−4	.	−78	3	3	−6	−6	3	−6	12	3	−6	4	.	2	2	−1	−1	2	−4	2	.	−1	.	.	2	.
$X.20$	168	8	.	168	6	6	6	6	6	−3	6	−3	6	.	.	8	2	2	2	−1	2	2	.	−1	.	.	.	.
$X.21$	168	8	.	168	6	6	6	6	6	−3	6	−3	6	.	.	8	2	2	2	−1	2	2	.	−1	.	.	.	.
$X.22$	182	−10	.	−91	−10	−10	20	20	−10	2	2	−1	−1	6	.	5	−4	2	2	2	2	−1	.	−1	.	.	.	−2
$X.23$	182	6	.	182	11	11	11	2	2	8	−4	8	−4	2	.	6	3	3	3	.	.	.	.	.	.	.	.	.
$X.24$	182	6	.	−91	A	$/A$	11	2	−1	8	−4	−4	2	2	.	−3	3	F	$/F$	.	.	.	.	.	.	.	.	.
$X.25$	182	6	.	−91	$/A$	A	11	2	−1	8	−4	−4	2	2	.	−3	3	$/F$	F	.	.	.	.	.	.	.	.	.
$X.26$	208	16	.	−104	−14	−14	28	10	−5	4	−2	−2	1	.	.	−8	4	−2	−2	4	−2	1	.	−2	.	.	−2	.
$X.27$	336	16	.	−168	−6	−6	12	12	−6	−6	12	3	−6	.	.	−8	4	−2	−2	−2	4	−2	.	1	.	.	.	.
$X.28$	364	12	.	−182	B	$/B$	13	−14	7	4	4	−2	−2	4	.	−6	−3	$-F$	$-/F$	.	.	.	.	.	.	.	.	.
$X.29$	364	12	.	−182	$/B$	B	13	−14	7	4	4	−2	−2	4	.	−6	−3	$-/F$	$-F$	.	.	.	.	.	.	.	.	.
$X.30$	364	12	.	364	13	13	13	−14	−14	4	4	4	4	4	.	12	−3	−3	−3	.	.	.	.	.	.	.	.	.
$X.31$	546	−14	.	546	33	33	33	6	6	6	6	6	6	−2	.	−14	1	1	1	−2	−2	−2	.	−2	.	.	.	.
$X.32$	546	−14	.	−273	C	$/C$	33	6	−3	6	6	−3	−3	−2	.	7	1	G	$/G$	−2	−2	1	.	1	.	.	.	.
$X.33$	546	−14	.	−273	$/C$	C	33	6	−3	6	6	−3	−3	−2	.	7	1	$/G$	G	−2	−2	1	.	1	.	.	.	.
$X.34$	729	9	−27	729	.	.	.	.	.	.	.	.	.	−3	−3	9	.	.	.	.	.	.	.	.	.	.	1	−1
$X.35$	729	9	27	729	.	.	.	.	.	.	.	.	.	−3	3	9	.	.	.	.	.	.	.	.	.	.	1	−1
$X.36$	819	3	21	819	9	9	9	9	9	.	.	.	.	−1	−3	3	−3	−3	−3	.	.	.	3	.	.	.	.	1
$X.37$	819	3	−21	819	9	9	9	9	9	.	.	.	.	−1	3	3	−3	−3	−3	.	.	.	−3	.	.	.	.	1
$X.38$	832	.	8	832	−32	−32	−32	−5	−5	4	4	4	4	.	.	.	.	.	.	.	.	.	−1	.	2	2	−1	.
$X.39$	832	.	−8	832	−32	−32	−32	−5	−5	4	4	4	4	.	.	.	.	.	.	.	.	.	1	.	−2	−2	−1	.
$X.40$	896	.	.	−448	−16	−16	32	−22	11	−4	−4	2	2	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.41$	896	.	.	−448	−16	−16	32	−22	11	−4	−4	2	2	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.42$	896	.	.	896	32	32	32	−22	−22	−4	−4	−4	−4	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.43$	1092	4	.	−546	D	$/D$	−15	12	−6	−6	−6	3	3	4	.	−2	1	$/G$	G	−2	−2	1	.	1	.	.	.	.
$X.44$	1092	4	.	−546	$/D$	D	−15	12	−6	−6	−6	3	3	4	.	−2	1	G	$/G$	−2	−2	1	.	1	.	.	.	.
$X.45$	1092	4	.	1092	−15	−15	−15	12	12	−6	−6	−6	−6	4	.	4	1	1	1	−2	−2	−2	.	−2	.	.	.	.
$X.46$	1456	−16	.	−728	E	$/E$	−2	−2	1	−2	−2	1	1	.	.	8	2	H	$/H$	2	2	−1	.	−1	.	.	.	.
$X.47$	1456	−16	.	−728	$/E$	E	−2	−2	1	−2	−2	1	1	.	.	8	2	$/H$	H	2	2	−1	.	−1	.	.	.	.
$X.48$	1456	−16	.	1456	−2	−2	−2	−2	−2	−2	−2	−2	−2	.	.	−16	2	2	2	2	2	2	.	2	.	.	.	.
$X.49$	1458	18	.	−729	.	.	.	.	.	.	.	.	.	−6	.	−9	.	.	.	.	.	.	.	.	.	.	2	−2
$X.50$	1638	6	.	−819	−9	−9	18	18	−9	.	.	.	.	−2	.	−3	−6	3	3	.	.	.	.	.	.	.	.	2
$X.51$	1664	.	.	−832	32	32	−64	−10	5	8	8	−4	−4	.	.	.	.	.	.	.	.	.	.	.	.	.	−2	.

(*continued on next page*)

Table 10.6 (*continued*)

2	1	1	1	.	.	.	5	5	2	2	2	2	2	.	1	1	1	1	.	3	3	.	.
3	4	4	4	4	4	4	2	2	2	2	2	1	1	1	.	2	2	2	1	1	1	1	1
7	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	1	.	.	.	.
13	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	1	1
	9a	9b	9c	9d	9e	9f	12a	12b	12c	12d	12e	12f	12g	13a	14a	18a	18b	18c	21a	24a	24b	39a	39b
2P	9a	9c	9b	9e	9d	9f	6a	6a	6c	6b	6d	6e	6e	13a	7a	9b	9c	9a	21a	12b	12a	39a	39b
3P	3e	3e	3e	3e	3e	3e	4a	4a	4a	4a	4a	4b	4b	13a	14a	6h	6h	6h	7a	8a	8a	13a	13a
7P	9a	9b	9c	9d	9e	9f	12a	12b	12c	12d	12e	12g	12f	13a	2b	18a	18b	18c	3a	24a	24b	39b	39a
13P	9a	9b	9c	9d	9e	9f	12a	12b	12c	12d	12e	12f	12g	1a	14a	18a	18b	18c	21a	24a	24b	3a	3a
X.1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.2	1	1	1	1	1	1	1	1	1	1	1	−1	−1	1	−1	−1	−1	−1	1	1	1	1	1
X.3	2	2	2	−1	−1	−1	−1	−1	−1	2	−1	.	.	2	.	.	.	.	−1	−1	−1	−1	−1
X.4	2	−1	−1	−1	−1	2	2	2	−1	−1	−1	.	.	1	.	−I	I	.	.	.	.	1	1
X.5	2	−1	−1	−1	−1	2	2	2	−1	−1	−1	.	.	1	.	I	−I	.	.	.	.	1	1
X.6	4	−2	−2	1	1	−2	−2	−2	1	−2	1	.	.	2	.	.	.	.	.	.	.	−1	−1
X.7	1	G	/G	G	/G	1	.	.	.	.	.	.	.	−1	1	/M	M	−1	1	.	.	−1	−1
X.8	1	/G	G	/G	G	1	.	.	.	.	.	.	.	−1	1	M	/M	−1	1	.	.	−1	−1
X.9	1	/G	G	/G	G	1	.	.	.	.	.	.	.	−1	−1	−M	−/M	1	1	.	.	−1	−1
X.10	1	G	/G	G	/G	1	.	.	.	.	.	.	.	−1	−1	−/M	−M	1	1	.	.	−1	−1
X.11	.	.	.	.	.	.	2	2	−1	−1	−1	−1	−1	.	1	.	.	.	1	.	.	.	.
X.12	.	.	.	.	.	.	2	2	−1	−1	−1	1	1	.	−1	.	.	.	1	.	.	.	.
X.13	1	1	1	1	1	1	3	3	.	.	.	1	1	.	.	−1	−1	−1	.	−1	−1	.	.
X.14	1	1	1	1	1	1	3	3	.	.	.	−1	−1	.	.	1	1	1	.	−1	−1	.	.
X.15	2	−1	−1	−1	−1	2	.	.	.	.	.	.	.	.	−1	1	1	−2	−1	.	.	.	.
X.16	2	−1	−1	−1	−1	2	.	.	.	.	.	.	.	.	1	−1	−1	2	−1	.	.	.	.
X.17	2	/H	H	−/G	−G	−1	.	.	.	.	.	.	.	−2	.	.	.	.	−1	.	.	1	1
X.18	2	H	/H	−G	−/G	−1	.	.	.	.	.	.	.	−2	.	.	.	.	−1	.	.	1	1
X.19	.	.	.	.	.	.	−2	−2	1	−2	1	.	.	.	.	.	.	.	−1	.	.	.	.
X.20	.	.	.	.	.	.	.	.	.	.	.	N	−N	−1	.	.	.	.	.	.	.	−1	−1
X.21	.	.	.	.	.	.	.	.	.	.	.	−N	N	−1	.	.	.	.	.	.	.	−1	−1
X.22	2	2	2	−1	−1	−1	−3	−3	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.
X.23	−4	2	2	2	2	−4	2	2	−1	−1	−1	.	.	.	.	.	.	.	.	.	.	.	.
X.24	−4	2	2	−1	−1	2	K	/K	M	−1	/M	.	.	.	.	.	.	.	.	−I	I	.	.
X.25	−4	2	2	−1	−1	2	/K	K	/M	−1	M	.	.	.	.	.	.	.	.	I	−I	.	.
X.26	4	−2	−2	1	1	−2	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.
X.27	.	.	.	.	.	.	.	.	.	.	.	.	.	−2	.	.	.	.	.	.	.	1	1
X.28	−2	−2	−2	1	1	1	−2	−2	G	1	/G	.	.	.	.	.	.	.	.	.	.	.	.
X.29	−2	−2	−2	1	1	1	−2	−2	/G	1	G	.	.	.	.	.	.	.	.	.	.	.	.
X.30	−2	−2	−2	−2	−2	−2	4	4	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.
X.31	.	.	.	.	.	.	−2	−2	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.
X.32	.	.	.	.	.	.	−/K	−K	−/M	1	−M	.	.	.	.	.	.	.	.	I	−I	.	.
X.33	.	.	.	.	.	.	−K	−/K	−M	1	−/M	.	.	.	.	.	.	.	.	−I	I	.	.
X.34	.	.	.	.	.	.	−3	−3	.	.	.	.	.	1	1	.	.	.	1	−1	−1	1	1
X.35	.	.	.	.	.	.	−3	−3	.	.	.	.	.	1	−1	.	.	.	1	−1	−1	1	1
X.36	.	.	.	.	.	.	−1	−1	−1	−1	−1	.	.	.	.	.	.	.	.	1	1	.	.
X.37	.	.	.	.	.	.	−1	−1	−1	−1	−1	.	.	.	.	.	.	.	.	1	1	.	.
X.38	1	1	1	1	1	1	.	.	.	.	.	.	.	.	1	−1	−1	−1	−1	.	.	.	.
X.39	1	1	1	1	1	1	.	.	.	.	.	.	.	.	−1	1	1	1	−1	.	.	.	.
X.40	2	2	2	−1	−1	−1	.	.	.	.	.	.	.	−1	.	.	.	.	.	.	.	O	/O
X.41	2	2	2	−1	−1	−1	.	.	.	.	.	.	.	−1	.	.	.	.	.	.	.	/O	O
X.42	2	2	2	2	2	2	.	.	.	.	.	.	.	−1	.	.	.	.	.	.	.	−1	−1
X.43	.	.	.	.	.	.	L	/L	−M	1	−/M	.	.	.	.	.	.	.	.	.	.	.	.
X.44	.	.	.	.	.	.	/L	L	−/M	1	−M	.	.	.	.	.	.	.	.	.	.	.	.
X.45	.	.	.	.	.	.	4	4	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.
X.46	−2	−2	−2	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
X.47	−2	−2	−2	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
X.48	−2	−2	−2	−2	−2	−2	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
X.49	.	.	.	.	.	.	3	3	.	.	.	.	.	2	.	.	.	.	−1	1	1	−1	−1
X.50	.	.	.	.	.	.	1	1	1	−2	1	.	.	.	.	.	.	.	.	−1	−1	.	.
X.51	2	2	2	−1	−1	−1	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.

Notes. Here $A = -\frac{11}{2} + \frac{27}{2}i\sqrt{3}$, $B = -\frac{13}{2} - \frac{27}{2}i\sqrt{3}$, $C = -\frac{33}{2} + \frac{27}{2}i\sqrt{3}$, $D = \frac{15}{2} - \frac{27}{2}i\sqrt{3}$, $E = 1 - 27i\sqrt{3}$, $F = \frac{3}{2}(-1 + i\sqrt{3})$, $G = -\frac{1}{2} + \frac{3}{2}i\sqrt{3}$, $H = -1 + 3i\sqrt{3}$, $I = i\sqrt{3}$, $J = -1 - i\sqrt{3}$, $K = -1 - 2i\sqrt{3}$, $L = -2 - 4i\sqrt{3}$, $M = \frac{1}{2}(1 + i\sqrt{3})$, $N = \sqrt{3}$ and $O = \frac{1}{2}(1 + i\sqrt{39})$.

Table 10.7
Character table of $N_4 = N_G(d_4)$

2	5	5	4	4	1	.	2	2	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	4
3	7	3	3	7	7	7	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	1
5	1	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
	1a	2a	2b	3a	3b	3c	3d	3e	3f	3g	3h	3i	3j	3k	3l	3m	3n	3o	3p	3q	3r	3s	3t	4a
2P	1a	1a	1a	3a	3b	3c	3d	3e	3h	3g	3f	3i	3q	3t	3l	3o	3n	3m	3p	3j	3s	3r	3k	2a
3P	1a	2a	2b	1a	1a	1a	1a	1a	1a	1a	1a	1a	1a	1a	1a	1a	1a	1a	1a	1a	1a	1a	1a	4a
5P	1a	2a	2b	3a	3b	3c	3d	3e	3h	3g	3f	3i	3q	3t	3l	3o	3n	3m	3p	3j	3s	3r	3k	4a
X.1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.2	1	1	−1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	−1
X.3	2	2	.	−1	2	−1	2	2	2	−1	2	−1	2	−1	2	−1	−1	−1	−1	2	−1	−1	−1	.
X.4	4	−4	.	4	4	4	−2	1	−2	−2	−2	1	1	1	−2	−2	−2	−2	−2	1	1	1	1	.
X.5	4	−4	.	4	4	4	−2	1	−2	−2	−2	1	1	1	−2	−2	−2	−2	−2	1	1	1	1	.
X.6	4	−4	.	4	4	4	1	−2	1	1	1	−2	−2	−2	1	1	1	1	1	−2	−2	−2	−2	.
X.7	4	−4	.	4	4	4	1	−2	1	1	1	−2	−2	−2	1	1	1	1	1	−2	−2	−2	−2	.
X.8	5	5	−1	5	5	5	−1	2	−1	−1	−1	2	2	2	−1	−1	−1	−1	−1	2	2	2	2	3
X.9	5	5	1	5	5	5	−1	2	−1	−1	−1	2	2	2	−1	−1	−1	−1	−1	2	2	2	2	−3
X.10	5	5	3	5	5	5	2	−1	2	2	2	−1	−1	−1	2	2	2	2	2	−1	−1	−1	−1	−1
X.11	5	5	−3	5	5	5	2	−1	2	2	2	−1	−1	−1	2	2	2	2	2	−1	−1	−1	−1	1
X.12	8	−8	.	−4	8	−4	−4	2	−4	2	−4	−1	2	−1	−4	2	2	2	2	2	−1	−1	−1	.
X.13	8	−8	.	−4	8	−4	2	−4	2	−1	2	2	−4	2	2	−1	−1	−1	−1	−4	2	2	2	.
X.14	9	9	3	9	9	9	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	3
X.15	9	9	−3	9	9	9	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−3
X.16	10	10	.	−5	10	−5	−2	4	−2	1	−2	−2	4	−2	−2	1	1	1	1	4	−2	−2	−2	.
X.17	10	10	.	−5	10	−5	4	−2	4	−2	4	1	−2	1	4	−2	−2	−2	−2	−2	1	1	1	.
X.18	10	10	−2	10	10	10	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2
X.19	10	10	2	10	10	10	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	−2
X.20	16	−16	.	−8	16	−8	−2	−2	−2	1	−2	1	−2	1	−2	1	1	1	1	−2	1	1	1	.
X.21	16	−16	.	−8	16	−8	−2	−2	−2	1	−2	1	−2	1	−2	1	1	1	1	−2	1	1	1	.
X.22	16	16	.	−8	16	−8	−2	−2	−2	1	−2	1	−2	1	−2	1	1	1	1	−2	1	1	1	.
X.23	16	16	.	−8	16	−8	−2	−2	−2	1	−2	1	−2	1	−2	1	1	1	1	−2	1	1	1	.
X.24	16	16	.	16	16	16	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	.
X.25	16	−16	.	16	16	16	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	.
X.26	18	18	.	−9	18	−9	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
X.27	20	−20	.	−10	20	−10	2	2	2	−1	2	−1	2	−1	2	−1	−1	−1	−1	2	−1	−1	−1	.
X.28	20	−20	.	−10	20	−10	2	2	2	−1	2	−1	2	−1	2	−1	−1	−1	−1	2	−1	−1	−1	.
X.29	20	20	.	−10	20	−10	2	2	2	−1	2	−1	2	−1	2	−1	−1	−1	−1	2	−1	−1	−1	.
X.30	20	−20	.	20	20	20	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	.
X.31	80	.	8	80	−1	−1	8	8	−1	8	−1	8	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	.
X.32	80	.	−8	80	−1	−1	8	8	−1	8	−1	8	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	.
X.33	80	.	8	80	−1	−1	2	−4	A	2	/A	−4	D	D	2	/A	2	A	2	/D	/D	D	/D	.
X.34	80	.	−8	80	−1	−1	2	−4	A	2	/A	−4	D	D	2	/A	2	A	2	/D	/D	D	/D	.
X.35	80	.	8	80	−1	−1	2	−4	/A	2	A	−4	/D	/D	2	A	2	/A	2	D	D	/D	D	.
X.36	80	.	−8	80	−1	−1	2	−4	/A	2	A	−4	/D	/D	2	A	2	/A	2	D	D	/D	D	.
X.37	160	.	.	160	−2	−2	−8	4	B	−8	/B	4	−/D	−/D	1	/B	1	B	1	−D	−D	−/D	−D	.
X.38	160	.	.	160	−2	−2	−8	4	/B	−8	B	4	−D	−D	1	B	1	/B	1	−/D	−/D	−D	−/D	.
X.39	160	.	.	−80	−2	1	16	16	−2	−8	−2	−8	−2	1	−2	1	1	1	1	−2	1	1	1	.
X.40	160	.	.	−80	−2	1	4	−8	4	−2	4	4	1	−5	−5	−2	7	−2	−2	1	4	4	−5	.
X.41	160	.	.	−80	−2	1	4	−8	4	−2	4	4	1	4	−5	−2	−2	−2	7	1	−5	−5	4	.
X.42	160	.	.	−80	−2	1	4	−8	C	−2	/C	4	B	−D	4	−/A	−2	−A	−2	/B	−/D	−D	−/D	.
X.43	160	.	.	−80	−2	1	4	−8	/C	−2	C	4	/B	−/D	4	−A	−2	−/A	−2	B	−D	−/D	−D	.
X.44	160	.	.	−80	−2	1	−8	4	/B	4	B	−2	−D	−/A	1	−D	4	−/D	−5	−/D	/E	E	−A	.
X.45	160	.	.	−80	−2	1	−8	4	/B	4	B	−2	−D	E	1	−D	−5	−/D	4	−/D	−A	−/A	/E	.
X.46	160	.	.	−80	−2	1	−8	4	B	4	/B	−2	−/D	−A	1	−/D	4	−D	−5	−D	E	/E	−/A	.
X.47	160	.	.	−80	−2	1	−8	4	B	4	/B	−2	−/D	/E	1	−/D	−5	−D	4	−D	−/A	−A	E	.
X.48	160	.	.	160	−2	−2	4	−8	4	4	4	−8	1	1	−5	4	−5	4	−5	1	1	1	1	.

(*continued on next page*)

Table 10.7 (*continued*)

2	4	1	4	2	2	1	1	1	2	2	1	1	3	3	1	3	2	2	1	1	3	3	1	1
3	1	1	3	3	3	3	3	3	2	2	2	2	1	.	1	1	1	1	1	1	1	1	1	1
5	.	1	1	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	1	1	.	.	1	1
	4b	5a	6a	6b	6c	6d	6e	6f	6g	6h	6i	6j	8a	8b	10a	12a	12b	12c	15a	15b	24a	24b	30a	30b
2P	2a	5a	3a	3e	3d	3b	3i	3g	3d	3d	3h	3f	4b	4b	5a	6a	6b	6b	15a	15b	12a	12a	15a	15b
3P	4b	5a	2a	2a	2a	2b	2a	2a	2b	2b	2b	2b	8a	8b	10a	4b	4a	4a	5a	5a	8a	8a	10a	10a
5P	4b	1a	6a	6b	6c	6d	6e	6f	6h	6g	6j	6i	8a	8b	2a	12a	12c	12b	3a	3a	24a	24b	6a	6a
X.1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.2	1	1	1	1	1	−1	1	1	−1	−1	−1	−1	1	−1	1	1	−1	−1	1	1	1	1	1	1
X.3	2	2	−1	2	2	.	−1	−1	.	.	.	.	2	.	2	−1	.	.	−1	−1	−1	−1	−1	−1
X.4	.	−1	−4	−1	2	.	−1	2	.	.	.	.	.	.	1	.	I	−I	−1	−1	.	.	1	1
X.5	.	−1	−4	−1	2	.	−1	2	.	.	.	.	.	.	1	.	−I	I	−1	−1	.	.	1	1
X.6	.	−1	−4	2	−1	.	2	−1	F	−F	−F	F	.	.	1	.	.	.	−1	−1	.	.	1	1
X.7	.	−1	−4	2	−1	.	2	−1	−F	F	F	−F	.	.	1	.	.	.	−1	−1	.	.	1	1
X.8	1	.	5	2	−1	−1	2	−1	−1	−1	−1	−1	−1	1	.	1	.	.	.	.	−1	−1	.	.
X.9	1	.	5	2	−1	1	2	−1	1	1	1	1	−1	−1	.	1	.	.	.	.	−1	−1	.	.
X.10	1	.	5	−1	2	3	−1	2	.	.	.	.	−1	1	.	1	−1	−1	.	.	−1	−1	.	.
X.11	1	.	5	−1	2	−3	−1	2	.	.	.	.	−1	−1	.	1	1	1	.	.	−1	−1	.	.
X.12	.	−2	4	−2	4	.	1	−2	.	.	.	.	.	.	2	.	.	.	1	1	.	.	−1	−1
X.13	.	−2	4	4	−2	.	−2	1	.	.	.	.	.	.	2	.	.	.	1	1	.	.	−1	−1
X.14	1	−1	9	.	.	3	.	.	.	.	.	.	1	−1	−1	1	.	.	−1	−1	1	1	−1	−1
X.15	1	−1	9	.	.	−3	.	.	.	.	.	.	1	1	−1	1	.	.	−1	−1	1	1	−1	−1
X.16	2	.	−5	4	−2	.	−2	1	.	.	.	.	−2	.	.	−1	.	.	.	.	1	1	.	.
X.17	2	.	−5	−2	4	.	1	−2	.	.	.	.	−2	.	.	−1	.	.	.	.	1	1	.	.
X.18	−2	.	10	1	1	−2	1	1	1	1	1	1	.	.	.	−2	−1	−1	.	.	.	.	.	.
X.19	−2	.	10	1	1	2	1	1	−1	−1	−1	−1	.	.	.	−2	1	1	.	.	.	.	.	.
X.20	.	1	8	2	2	.	−1	−1	.	.	.	.	.	.	−1	.	.	.	J	/J	.	.	−J	−/J
X.21	.	1	8	2	2	.	−1	−1	.	.	.	.	.	.	−1	.	.	.	/J	J	.	.	−/J	−J
X.22	.	1	−8	−2	−2	.	1	1	.	.	.	.	.	.	1	.	.	.	J	/J	.	.	J	/J
X.23	.	1	−8	−2	−2	.	1	1	.	.	.	.	.	.	1	.	.	.	/J	J	.	.	/J	J
X.24	.	1	16	−2	−2	.	−2	−2	.	.	.	.	.	.	1	.	.	.	1	1	.	.	1	1
X.25	.	1	−16	2	2	.	2	2	.	.	.	.	.	.	−1	.	.	.	1	1	.	.	−1	−1
X.26	2	−2	−9	.	.	.	.	.	.	.	.	.	2	.	−2	−1	.	.	1	1	−1	−1	1	1
X.27	.	.	10	−2	−2	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	K	−K	.	.
X.28	.	.	10	−2	−2	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	−K	K	.	.
X.29	−4	.	−10	2	2	.	−1	−1	.	.	.	.	.	.	.	2	.	.	.	.	.	.	.	.
X.30	.	.	−20	−2	−2	.	−2	−2	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
X.31	.	.	.	.	.	−1	.	.	2	2	−1	−1	.	.	.	.	.	.	.	.	.	.	.	.
X.32	.	.	.	.	.	1	.	.	−2	−2	1	1	.	.	.	.	.	.	.	.	.	.	.	.
X.33	.	.	.	.	.	−1	.	.	G	/G	H	/H	.	.	.	.	.	.	.	.	.	.	.	.
X.34	.	.	.	.	.	1	.	.	−G	−/G	−H	−/H	.	.	.	.	.	.	.	.	.	.	.	.
X.35	.	.	.	.	.	−1	.	.	/G	G	/H	H	.	.	.	.	.	.	.	.	.	.	.	.
X.36	.	.	.	.	.	1	.	.	−/G	−G	−/H	−H	.	.	.	.	.	.	.	.	.	.	.	.
X.37	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
X.38	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
X.39	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
X.40	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
X.41	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
X.42	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
X.43	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
X.44	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
X.45	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
X.46	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
X.47	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
X.48	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

Notes. Here $A = -\frac{1}{2}(5+3i\sqrt{3})$, $B = 1+3i\sqrt{3}$, $C = -5-3i\sqrt{3}$, $D = \frac{1}{2}(1+3i\sqrt{3})$, $E = -2+3i\sqrt{3}$, $F = -i\sqrt{3}$, $G = -1-i\sqrt{3}$, $H = \frac{1}{2}(1-i\sqrt{3})$, $I = \sqrt{3}$, $J = -\frac{1}{2}(1-i\sqrt{15})$ and $K = -i\sqrt{6}$.

Table 10.8

Character table of $N_1 = N_G(D_1)$

2	4	4	2	1	3	1	2	1	1	1	1	.	1	.	3	3	1	1	2	2	2	1	1	1	1	1
3	10	4	4	10	7	8	7	6	6	6	5	5	4	4	1	3	4	4	3	3	3	3	3	3	2	2
	1a	2a	2b	3a	3b	3c	3d	3e	3f	3g	3h	3i	3j	3k	4a	6a	6b	6c	6d	6e	6f	6g	6h	6i	6j	6k
2P	1a	1a	1a	3a	3b	3c	3d	3e	3f	3g	3h	3i	3j	3k	2a	3b	3c	3a	3d	3d	3d	3g	3f	3e	3j	3h
3P	1a	2a	2b	1a	1a	1a	1a	1a	1a	1a	1a	1a	1a	1a	4a	2a	2a	2b	2a	2b	2b	2b	2b	2b	2a	2b
$X.1$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$X.2$	1	1	−1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	−1	1	−1	−1	−1	−1	−1	1	−1
$X.3$	2	−2	.	2	2	2	2	2	2	2	2	2	−1	−1	.	−2	−2	.	−2	.	.	.	.	.	1	.
$X.4$	2	−2	.	2	2	2	2	2	2	2	2	2	−1	−1	.	−2	−2	.	−2	.	.	.	.	.	1	.
$X.5$	2	2	.	2	2	2	2	2	2	2	2	2	−1	−1	2	2	2	.	2	.	.	.	.	.	−1	.
$X.6$	3	3	1	3	3	3	3	3	3	3	3	3	.	.	−1	3	3	1	3	1	1	1	1	1	.	1
$X.7$	3	3	−1	3	3	3	3	3	3	3	3	3	.	.	−1	3	3	−1	3	−1	−1	−1	−1	−1	.	−1
$X.8$	4	−4	.	4	4	4	4	4	4	4	4	4	1	1	.	−4	−4	.	−4	.	.	.	.	.	−1	.
$X.9$	8	.	2	8	8	8	8	5	5	5	−1	−1	2	−1	.	.	.	2	.	2	2	−1	−1	−1	.	−1
$X.10$	8	.	−2	8	8	8	8	5	5	5	−1	−1	2	−1	.	.	.	−2	.	−2	−2	1	1	1	.	1
$X.11$	8	.	2	8	8	8	8	−1	−1	−1	−1	−1	2	2	.	.	.	2	.	2	2	−1	−1	−1	.	−1
$X.12$	8	.	−2	8	8	8	8	−1	−1	−1	−1	−1	2	2	.	.	.	−2	.	−2	−2	1	1	1	.	1
$X.13$	16	.	.	16	16	16	16	10	10	10	−2	−2	−2	1	.	.	.	.	.	.	.	.	.	.	.	.
$X.14$	16	.	.	16	16	16	16	−8	−8	−8	−2	−2	−2	1	.	.	.	.	.	.	.	.	.	.	.	.
$X.15$	16	.	.	16	16	16	16	−8	−8	−8	−2	−2	−2	1	.	.	.	.	.	.	.	.	.	.	.	.
$X.16$	16	.	.	16	16	16	16	−2	−2	−2	−2	−2	−2	−2	.	.	.	.	.	.	.	.	.	.	.	.
$X.17$	16	.	.	16	16	16	16	−8	−8	−8	−2	−2	4	−2	.	.	.	.	.	.	.	.	.	.	.	.
$X.18$	24	8	.	24	−12	−3	6	.	.	.	6	−3	−3	.	.	−4	−1	.	2	.	.	.	.	.	−1	.
$X.19$	24	8	.	24	−12	−3	6	.	.	.	6	−3	.	3	.	−4	−1	.	2	.	.	.	.	.	2	.
$X.20$	24	8	.	24	−12	−3	6	.	.	.	6	−3	3	−3	.	−4	−1	.	2	.	.	.	.	.	−1	.
$X.21$	24	−8	.	24	−12	−3	6	.	.	.	6	−3	−3	.	.	4	1	.	−2	.	.	.	.	.	1	.
$X.22$	24	−8	.	24	−12	−3	6	.	.	.	6	−3	.	3	.	4	1	.	−2	.	.	.	.	.	−2	.
$X.23$	24	−8	.	24	−12	−3	6	.	.	.	6	−3	3	−3	.	4	1	.	−2	.	.	.	.	.	1	.
$X.24$	24	.	−2	24	24	24	24	6	6	6	−3	−3	.	.	.	.	.	−2	.	−2	−2	−2	−2	−2	.	1
$X.25$	24	.	2	24	24	24	24	6	6	6	−3	−3	.	.	.	.	.	2	.	2	2	2	2	2	.	−1
$X.26$	24	.	−2	24	24	24	24	−3	−3	−3	6	6	.	.	.	.	.	−2	.	−2	−2	1	1	1	.	−2
$X.27$	24	.	2	24	24	24	24	−3	−3	−3	6	6	.	.	.	.	.	2	.	2	2	−1	−1	−1	.	2
$X.28$	48	.	.	48	−24	−6	12	.	.	.	−6	3	−6	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.29$	48	.	.	48	−24	−6	12	.	.	.	−6	3	.	−3	.	.	.	.	.	.	.	.	.	.	.	.
$X.30$	48	.	.	48	−24	−6	12	.	.	.	−6	3	6	3	.	.	.	.	.	.	.	.	.	.	.	.
$X.31$	54	6	.	54	27	−27	.	.	.	.	.	.	.	.	−2	3	−3	.	.	.	.	.	.	.	.	.
$X.32$	54	6	.	54	27	−27	.	.	.	.	.	.	.	.	2	3	−3	.	.	.	.	.	.	.	.	.
$X.33$	54	6	.	54	27	−27	.	.	.	.	.	.	.	.	2	3	−3	.	.	.	.	.	.	.	.	.
$X.34$	54	6	.	54	27	−27	.	.	.	.	.	.	.	.	−2	3	−3	.	.	.	.	.	.	.	.	.
$X.35$	54	−6	.	54	27	−27	.	.	.	.	.	.	.	.	.	−3	3	.	.	.	.	.	.	.	.	.
$X.36$	54	−6	.	54	27	−27	.	.	.	.	.	.	.	.	.	−3	3	.	.	.	.	.	.	.	.	.
$X.37$	54	−6	.	54	27	−27	.	.	.	.	.	.	.	.	.	−3	3	.	.	.	.	.	.	.	.	.
$X.38$	54	−6	.	54	27	−27	.	.	.	.	.	.	.	.	.	−3	3	.	.	.	.	.	.	.	.	.
$X.39$	108	12	6	108	.	27	−27	.	.	.	.	.	.	.	.	.	3	6	−3	−3	−3	.	.	.	.	.
$X.40$	108	12	−6	108	.	27	−27	.	.	.	.	.	.	.	.	.	3	−6	−3	3	3	.	.	.	.	.
$X.41$	108	−12	.	108	.	27	−27	.	.	.	.	.	.	.	.	.	−3	.	3	A	$-A$	.	.	.	.	.
$X.42$	108	−12	.	108	.	27	−27	.	.	.	.	.	.	.	.	.	−3	.	3	$-A$	A	.	.	.	.	.
$X.43$	144	.	.	144	−72	−18	36	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.44$	216	.	6	−27	.	.	.	−9	9	.	.	.	.	.	.	.	.	−3	.	.	.	.	3	−3	.	.
$X.45$	216	.	−6	−27	.	.	.	−9	9	.	.	.	.	.	.	.	.	3	.	.	.	.	−3	3	.	.
$X.46$	216	.	6	−27	.	.	.	.	−9	9	.	.	.	.	.	.	.	−3	.	.	.	3	−3	.	.	.
$X.47$	216	.	−6	−27	.	.	.	.	−9	9	.	.	.	.	.	.	.	3	.	.	.	−3	3	.	.	.
$X.48$	216	.	6	−27	.	.	.	9	.	−9	.	.	.	.	.	.	.	−3	.	.	.	−3	.	3	.	.
$X.49$	216	.	−6	−27	.	.	.	9	.	−9	.	.	.	.	.	.	.	3	.	.	.	3	.	−3	.	.
$X.50$	432	.	.	−54	.	.	.	−18	18	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.51$	432	.	.	−54	.	.	.	.	−18	18	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.52$	432	.	.	−54	.	.	.	18	.	−18	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

(*continued on next page*)

Table 10.8 (*continued*)

2	3	3	1	.	1	.	.	.	.	.	1	1	.	3	3	1	1	1	1	3	3	3	3	.	.	.
3	1	1	6	6	5	5	5	5	5	5	4	4	4	1	1	2	2	2	2	1	1	1	1	3	3	3
	$8a$	$8b$	$9a$	$9b$	$9c$	$9d$	$9e$	$9f$	$9g$	$9h$	$9i$	$9j$	$9k$	$12a$	$12b$	$18a$	$18b$	$18c$	$18d$	$24a$	$24b$	$24c$	$24d$	$27a$	$27b$	$27c$
$2P$	$4a$	$4a$	$9a$	$9b$	$9c$	$9d$	$9e$	$9f$	$9g$	$9h$	$9i$	$9j$	$9k$	$6a$	$6a$	$9c$	$9j$	$9a$	$9i$	$12b$	$12a$	$12b$	$12a$	$27a$	$27c$	$27b$
$3P$	$8a$	$8b$	$3a$	$3a$	$3c$	$3a$	$3c$	$3c$	$3c$	$3c$	$3a$	$3c$	$3c$	$4a$	$4a$	$6b$	$6b$	$6c$	$6c$	$8a$	$8a$	$8b$	$8b$	$9b$	$9b$	$9b$
$X.1$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$X.2$	−1	−1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	1	1	1
$X.3$	B	$-B$	2	2	−1	2	−1	−1	−1	−1	2	−1	−1	.	.	1	1	.	.	B	B	$-B$	$-B$	−1	−1	−1
$X.4$	$-B$	B	2	2	−1	2	−1	−1	−1	−1	2	−1	−1	.	.	1	1	.	.	$-B$	$-B$	B	B	−1	−1	−1
$X.5$	.	.	2	2	−1	2	−1	−1	−1	−1	2	−1	−1	2	2	−1	−1	.	.	.	.	.	.	−1	−1	−1
$X.6$	−1	−1	3	3	.	3	.	.	.	.	3	.	.	−1	−1	.	.	1	1	−1	−1	−1	−1	.	.	.
$X.7$	1	1	3	3	.	3	.	.	.	.	3	.	.	−1	−1	.	.	−1	−1	1	1	1	1	.	.	.
$X.8$	.	.	4	4	1	4	1	1	1	1	4	1	1	.	.	−1	−1	.	.	.	.	.	.	1	1	1
$X.9$	.	.	−1	−1	2	−4	2	−1	−1	−1	2	2	−1	.	.	.	.	−1	2	.	.	.	.	2	−1	−1
$X.10$	.	.	−1	−1	2	−4	2	−1	−1	−1	2	2	−1	.	.	.	.	1	−2	.	.	.	.	2	−1	−1
$X.11$	.	.	8	8	2	−1	2	2	2	2	−1	2	2	.	.	.	.	2	−1	.	.	.	.	−1	−1	−1
$X.12$	.	.	8	8	2	−1	2	2	2	2	−1	2	2	.	.	.	.	−2	1	.	.	.	.	−1	−1	−1
$X.13$	.	.	−2	−2	−2	−8	−2	1	1	1	4	−2	1	.	.	.	.	.	.	.	.	.	.	−2	1	1
$X.14$	.	.	−2	−2	−2	1	−2	1	1	1	4	−2	1	.	.	.	.	.	.	.	.	.	.	1	F	$/F$
$X.15$	.	.	−2	−2	−2	1	−2	1	1	1	4	−2	1	.	.	.	.	.	.	.	.	.	.	1	$/F$	F
$X.16$	.	.	16	16	−2	−2	−2	−2	−2	−2	−2	−2	−2	.	.	.	.	.	.	.	.	.	.	1	1	1
$X.17$	.	.	−2	−2	4	1	4	−2	−2	−2	4	4	−2	.	.	.	.	.	.	.	.	.	.	−2	1	1
$X.18$	.	.	6	−3	.	.	.	3	3	3	.	3	−3	.	.	2	−1	.	.	.	.	.	.	.	.	.
$X.19$	.	.	6	−3	3	.	3	−3	−3	−3	.	−3	.	.	.	−1	−1	.	.	.	.	.	.	.	.	.
$X.20$	.	.	6	−3	−3	.	−3	.	.	.	.	.	3	.	.	−1	2	.	.	.	.	.	.	.	.	.
$X.21$	.	.	6	−3	.	.	.	3	3	3	.	3	−3	.	.	−2	1	.	.	.	.	.	.	.	.	.
$X.22$	.	.	6	−3	3	.	3	−3	−3	−3	.	−3	.	.	.	1	1	.	.	.	.	.	.	.	.	.
$X.23$	.	.	6	−3	−3	.	−3	.	.	.	.	.	3	.	.	1	−2	.	.	.	.	.	.	.	.	.
$X.24$	.	.	−3	−3	.	6	.	.	.	.	−3	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.
$X.25$	.	.	−3	−3	.	6	.	.	.	.	−3	.	.	.	.	.	.	−1	−1	.	.	.	.	.	.	.
$X.26$	.	.	−3	−3	.	−3	.	.	.	.	−3	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.
$X.27$	.	.	−3	−3	.	−3	.	.	.	.	−3	.	.	.	.	.	.	−1	−1	.	.	.	.	.	.	.
$X.28$	.	.	12	−6	.	.	.	−3	−3	−3	.	6	3	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.29$	.	.	12	−6	6	.	6	3	3	3	.	−6	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.30$	.	.	12	−6	−6	.	−6	.	.	.	.	.	−3	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.31$	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	C	$-C$	C	$-C$	.	.	.
$X.32$	−2	−2	.	.	.	.	.	.	.	.	.	.	.	−1	−1	.	.	.	.	1	1	1	1	.	.	.
$X.33$	2	2	.	.	.	.	.	.	.	.	.	.	.	−1	−1	.	.	.	.	−1	−1	−1	−1	.	.	.
$X.34$	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	$-C$	C	$-C$	C	.	.	.
$X.35$	B	$-B$	.	.	.	.	.	.	.	.	.	.	.	C	$-C$	.	.	.	.	D	E	$-D$	$-E$	.	.	.
$X.36$	B	$-B$	.	.	.	.	.	.	.	.	.	.	.	$-C$	C	.	.	.	.	E	D	$-E$	$-D$	.	.	.
$X.37$	$-B$	B	.	.	.	.	.	.	.	.	.	.	.	$-C$	C	.	.	.	.	$-E$	$-D$	E	D	.	.	.
$X.38$	$-B$	B	.	.	.	.	.	.	.	.	.	.	.	C	$-C$	.	.	.	.	$-D$	$-E$	D	E	.	.	.
$X.39$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.40$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.41$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.42$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.43$	.	.	−18	9	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.44$	.	.	.	.	6	.	−3	−3	−3	6	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.45$	.	.	.	.	6	.	−3	−3	−3	6	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.46$	.	.	.	.	6	.	−3	6	−3	−3	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.47$	.	.	.	.	6	.	−3	6	−3	−3	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.48$	.	.	.	.	6	.	−3	−3	6	−3	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.49$	.	.	.	.	6	.	−3	−3	6	−3	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.50$	.	.	.	.	−6	.	3	3	3	−6	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.51$	.	.	.	.	−6	.	3	−6	3	3	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.52$	.	.	.	.	−6	.	3	3	−6	3	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

Notes. Here $A = -3i\sqrt{3}$, $B = -i\sqrt{2}$, $C = \sqrt{3}$, $D = -\xi^{17} - \xi^{19}$, $E = -\xi - \xi^{11}$ and $F = -\frac{1}{2}(1 - 3i\sqrt{3})$, where ξ is a 24th root of unity.

Table 10.9

Character table of $N_{10} = N_G(d_{10})$

2	4	4	2	3	1	4	4	1	1	1	1	1	1	1	1	.	.	.	.	.	3	3	4
3	10	4	4	10	9	7	7	7	7	7	5	5	5	5	5	5	5	5	5	5	3	4	3
	1a	2a	2b	3a	3b	3c	3d	3e	3f	3g	3h	3i	3j	3k	3l	3m	3n	3o	3p	3q	4a	6a	6b
2P	1a	1a	1a	3a	3b	3d	3c	3e	3f	3g	3h	3i	3j	3k	3l	3m	3n	3q	3p	3o	2a	3a	3d
3P	1a	2a	2b	1a	1a	1a	1a	1a	1a	1a	1a	1a	1a	1a	1a	1a	1a	1a	1a	1a	4a	2a	2a
X.1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.2	1	1	−1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.3	2	−2	.	2	2	2	2	2	2	2	2	−1	−1	−1	−1	2	−1	−1	−1	−1	.	−2	−2
X.4	2	−2	.	2	2	2	2	2	2	2	2	−1	−1	−1	−1	2	−1	−1	−1	−1	.	−2	−2
X.5	2	2	.	2	2	2	2	2	2	2	2	−1	−1	−1	−1	2	−1	−1	−1	−1	2	2	2
X.6	3	3	1	3	3	3	3	3	3	3	3	.	.	.	.	3	.	.	.	.	−1	3	3
X.7	3	3	−1	3	3	3	3	3	3	3	3	.	.	.	.	3	.	.	.	.	−1	3	3
X.8	4	−4	.	4	4	4	4	4	4	4	4	1	1	1	1	4	1	1	1	1	.	−4	−4
X.9	6	−2	.	6	6	6	6	6	6	6	.	.	.	.	3	.	3	−3	−3	−3	2	−2	−2
X.10	6	−2	.	6	6	6	6	6	6	6	.	3	3	3	−3	.	−3	.	.	.	2	−2	−2
X.11	6	−2	.	6	6	6	6	6	6	6	.	−3	−3	−3	.	.	.	3	3	3	2	−2	−2
X.12	8	.	2	8	8	8	8	8	8	8	−1	2	2	2	2	−1	2	2	2	2	.	.	.
X.13	8	.	−2	8	8	8	8	8	8	8	−1	2	2	2	2	−1	2	2	2	2	.	.	.
X.14	8	.	2	8	8	8	8	−1	−1	−1	5	2	2	2	2	−4	2	−1	−1	−1	.	.	.
X.15	8	.	−2	8	8	8	8	−1	−1	−1	5	2	2	2	2	−4	2	−1	−1	−1	.	.	.
X.16	12	4	.	12	12	12	12	12	12	12	.	.	.	.	−3	.	−3	3	3	3	.	4	4
X.17	12	4	.	12	12	12	12	12	12	12	.	3	3	3	.	.	.	−3	−3	−3	.	4	4
X.18	12	4	.	12	12	12	12	12	12	12	.	−3	−3	−3	3	.	3	.	.	.	.	4	4
X.19	16	.	.	16	16	16	16	16	16	16	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	.	.	.
X.20	16	.	.	16	16	16	16	−2	−2	−2	−8	−2	−2	−2	−2	1	−2	1	1	1	.	.	.
X.21	16	.	.	16	16	16	16	−2	−2	−2	−8	−2	−2	−2	−2	1	−2	1	1	1	.	.	.
X.22	16	.	.	16	16	16	16	−2	−2	−2	10	−2	−2	−2	−2	−8	−2	1	1	1	.	.	.
X.23	16	.	.	16	16	16	16	−2	−2	−2	−8	4	4	4	4	1	4	−2	−2	−2	.	.	.
X.24	18	−6	.	18	18	18	18	18	18	18	.	.	.	.	.	.	.	.	.	.	−2	−6	−6
X.25	24	.	−2	24	24	24	24	−3	−3	−3	6	.	.	.	.	6	.	.	.	.	.	.	.
X.26	24	.	2	24	24	24	24	−3	−3	−3	6	.	.	.	.	6	.	.	.	.	.	.	.
X.27	24	.	−2	24	24	24	24	−3	−3	−3	−3	.	.	.	.	−3	.	.	.	.	.	.	.
X.28	24	.	2	24	24	24	24	−3	−3	−3	−3	.	.	.	.	−3	.	.	.	.	.	.	.
X.29	27	3	−3	27	27	A	/A	.	.	.	.	.	.	.	.	.	.	.	.	.	3	3	D
X.30	27	3	3	27	27	A	/A	.	.	.	.	.	.	.	.	.	.	.	.	.	3	3	D
X.31	27	3	−3	27	27	/A	A	.	.	.	.	.	.	.	.	.	.	.	.	.	3	3	/D
X.32	27	3	3	27	27	/A	A	.	.	.	.	.	.	.	.	.	.	.	.	.	3	3	/D
X.33	27	3	3	27	27	/A	A	.	.	.	.	.	.	.	.	.	.	.	.	.	−1	3	/D
X.34	27	3	−3	27	27	/A	A	.	.	.	.	.	.	.	.	.	.	.	.	.	−1	3	/D
X.35	27	3	3	27	27	A	/A	.	.	.	.	.	.	.	.	.	.	.	.	.	−1	3	D
X.36	27	3	−3	27	27	A	/A	.	.	.	.	.	.	.	.	.	.	.	.	.	−1	3	D
X.37	48	.	.	48	48	48	48	−6	−6	−6	.	.	.	.	6	.	6	3	3	3	.	.	.
X.38	48	.	.	48	48	48	48	−6	−6	−6	.	6	6	6	−6	.	−6	.	.	.	.	.	.
X.39	48	.	.	48	48	48	48	−6	−6	−6	.	−6	−6	−6	.	.	.	−3	−3	−3	.	.	.
X.40	54	−6	.	54	54	B	/B	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−6	E
X.41	54	−6	.	54	54	B	/B	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−6	E
X.42	54	−6	.	54	54	/B	B	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−6	/E
X.43	54	−6	.	54	54	/B	B	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−6	/E
X.44	54	6	.	54	54	B	/B	.	.	.	.	.	.	.	.	.	.	.	.	.	−2	6	−E
X.45	54	6	.	54	54	/B	B	.	.	.	.	.	.	.	.	.	.	.	.	.	−2	6	−/E
X.46	72	.	6	72	−9	.	.	.	−9	9	.	.	.	.	6	.	−3	−3	6	−3	.	.	.
X.47	72	.	−6	72	−9	.	.	.	−9	9	.	.	.	.	6	.	−3	−3	6	−3	.	.	.
X.48	144	.	.	144	−18	.	.	.	−18	18	.	.	.	.	−6	.	3	3	−6	3	.	.	.
X.49	144	.	.	144	−18	.	.	.	−18	18	.	.	.	.	12	.	−6	3	−6	3	.	.	.
X.50	144	.	.	144	−18	.	.	.	−18	18	.	.	.	.	−6	.	3	C	3	/C	.	.	.
X.51	144	.	.	144	−18	.	.	.	−18	18	.	.	.	.	−6	.	3	/C	3	C	.	.	.
X.52	162	−6	.	−81	.	.	.	.	.	.	.	9	−9	.	.	.	.	.	.	.	6	3	.
X.53	162	−6	.	−81	.	.	.	.	.	.	.	−9	.	9	.	.	.	.	.	.	6	3	.
X.54	162	−6	.	−81	.	.	.	.	.	.	.	.	9	−9	.	.	.	.	.	.	6	3	.
X.55	216	.	6	216	−27	.	.	−27	27	.	.	.	.	.	.	.	.	.	.	.	.	.	.
X.56	216	.	−6	216	−27	.	.	−27	27	.	.	.	.	.	.	.	.	.	.	.	.	.	.
X.57	216	.	6	216	−27	.	.	27	.	−27	.	.	.	.	.	.	.	.	.	.	.	.	.
X.58	216	.	−6	216	−27	.	.	27	.	−27	.	.	.	.	.	.	.	.	.	.	.	.	.
X.59	324	12	.	−162	.	.	.	.	.	.	.	−9	9	.	.	.	.	.	.	.	.	−6	.
X.60	324	12	.	−162	.	.	.	.	.	.	.	.	−9	9	.	.	.	.	.	.	.	−6	.
X.61	324	12	.	−162	.	.	.	.	.	.	.	9	.	−9	.	.	.	.	.	.	.	−6	.
X.62	486	−18	.	−243	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−6	9	.

(*continued on next page*)

Table 10.9 (*continued*)

2	4	1	2	2	1	1	1	1	1	1	1	1	3	3	3	.	1	.	1	1	.	.	2	3	3
3	3	4	3	3	3	3	3	3	3	3	2	2	1	1	6	6	5	5	4	4	4	4	3	2	2
	$6c$	$6d$	$6e$	$6f$	$6g$	$6h$	$6i$	$6j$	$6k$	$6l$	$6m$	$6n$	$8a$	$8b$	$9a$	$9b$	$9c$	$9d$	$9e$	$9f$	$9g$	$9h$	$12a$	$12b$	$12c$
$2P$	$3c$	$3b$	$3d$	$3c$	$3i$	$3j$	$3k$	$3g$	$3e$	$3f$	$3l$	$3h$	$4a$	$4a$	$9a$	$9b$	$9c$	$9d$	$9e$	$9f$	$9g$	$9h$	$6a$	$6c$	$6b$
$3P$	$2a$	$2b$	$2b$	$2b$	$2a$	$2a$	$2a$	$2b$	$2b$	$2b$	$2a$	$2b$	$8a$	$8b$	$3a$	$3a$	$3b$	$3b$	$3b$	$3a$	$3b$	$3b$	$4a$	$4a$	$4a$
$X.1$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$X.2$	1	−1	−1	−1	1	1	1	−1	−1	−1	1	−1	−1	−1	1	1	1	1	1	1	1	1	1	1	1
$X.3$	−2	.	.	.	1	1	1	.	.	.	1	.	F	$-F$	2	2	2	2	2	−1	−1	−1	.	.	.
$X.4$	−2	.	.	.	1	1	1	.	.	.	1	.	$-F$	F	2	2	2	2	2	−1	−1	−1	.	.	.
$X.5$	2	.	.	.	−1	−1	−1	.	.	.	−1	.	.	.	2	2	2	2	2	−1	−1	−1	2	2	2
$X.6$	3	1	1	1	.	.	.	1	1	1	.	1	−1	−1	3	3	3	3	3	.	.	.	−1	−1	−1
$X.7$	3	−1	−1	−1	.	.	.	−1	−1	−1	.	−1	1	1	3	3	3	3	3	.	.	.	−1	−1	−1
$X.8$	−4	.	.	.	−1	−1	−1	.	.	.	−1	.	.	.	4	4	4	4	4	1	1	1	.	.	.
$X.9$	−2	.	.	.	−2	−2	−2	.	.	.	1	.	.	.	−3	−3	.	.	.	−3	.	3	2	2	2
$X.10$	−2	.	.	.	1	1	1	.	.	.	1	.	.	.	−3	−3	.	.	.	.	3	−3	2	2	2
$X.11$	−2	.	.	.	1	1	1	.	.	.	−2	.	.	.	−3	−3	.	.	.	3	−3	.	2	2	2
$X.12$	.	2	2	2	.	.	.	2	2	2	.	−1	.	.	8	8	−1	−1	−1	2	2	2	.	.	.
$X.13$	.	−2	−2	−2	.	.	.	−2	−2	−2	.	1	.	.	8	8	−1	−1	−1	2	2	2	.	.	.
$X.14$	.	2	2	2	.	.	.	−1	−1	−1	.	−1	.	.	8	−1	−1	−1	2	2	−1	−1	.	.	.
$X.15$	.	−2	−2	−2	.	.	.	1	1	1	.	1	.	.	8	−1	−1	−1	2	2	−1	−1	.	.	.
$X.16$	4	.	.	.	−2	−2	−2	.	.	.	1	.	.	.	−6	−6	.	.	.	3	.	−3	.	.	.
$X.17$	4	.	.	.	1	1	1	.	.	.	−2	.	.	.	−6	−6	.	.	.	−3	3	.	.	.	.
$X.18$	4	.	.	.	1	1	1	.	.	.	1	.	.	.	−6	−6	.	.	.	.	−3	3	.	.	.
$X.19$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	16	16	−2	−2	−2	−2	−2	−2	.	.	.
$X.20$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	16	−2	−2	−2	4	−2	1	1	.	.	.
$X.21$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	16	−2	−2	−2	4	−2	1	1	.	.	.
$X.22$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	16	−2	−2	−2	4	−2	1	1	.	.	.
$X.23$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	16	−2	−2	−2	4	4	−2	−2	.	.	.
$X.24$	−6	.	.	.	.	.	.	.	.	.	.	.	.	.	−9	−9	.	.	.	.	.	.	−2	−2	−2
$X.25$	.	−2	−2	−2	.	.	.	1	1	1	.	−2	.	.	24	−3	−3	−3	−3	.	.	.	.	.	.
$X.26$	.	2	2	2	.	.	.	−1	−1	−1	.	2	.	.	24	−3	−3	−3	−3	.	.	.	.	.	.
$X.27$	.	−2	−2	−2	.	.	.	1	1	1	.	1	.	.	24	−3	6	6	−3	.	.	.	.	.	.
$X.28$	.	2	2	2	.	.	.	−1	−1	−1	.	−1	.	.	24	−3	6	6	−3	.	.	.	.	.	.
$X.29$	$/D$	−3	$-D$	$-/D$	.	.	.	.	.	.	.	.	−1	−1	.	.	.	.	.	.	.	.	3	D	$/D$
$X.30$	$/D$	3	D	$/D$	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.	3	D	$/D$
$X.31$	D	−3	$-/D$	$-D$	.	.	.	.	.	.	.	.	−1	−1	.	.	.	.	.	.	.	.	3	$/D$	D
$X.32$	D	3	$/D$	D	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.	3	$/D$	D
$X.33$	D	3	$/D$	D	.	.	.	.	.	.	.	.	−1	−1	.	.	.	.	.	.	.	.	−1	G	$/G$
$X.34$	D	−3	$-/D$	$-D$	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.	−1	G	$/G$
$X.35$	$/D$	3	D	$/D$	.	.	.	.	.	.	.	.	−1	−1	.	.	.	.	.	.	.	.	−1	$/G$	G
$X.36$	$/D$	−3	$-D$	$-/D$	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.	−1	$/G$	G
$X.37$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−24	3	.	.	.	−6	.	−3	.	.	.
$X.38$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−24	3	.	.	.	.	−3	3	.	.	.
$X.39$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−24	3	.	.	.	6	3	.	.	.	.
$X.40$	$/E$	.	.	.	.	.	.	.	.	.	.	.	F	$-F$	.	.	.	.	.	.	.	.	.	.	.
$X.41$	$/E$	.	.	.	.	.	.	.	.	.	.	.	$-F$	F	.	.	.	.	.	.	.	.	.	.	.
$X.42$	E	.	.	.	.	.	.	.	.	.	.	.	F	$-F$	.	.	.	.	.	.	.	.	.	.	.
$X.43$	E	.	.	.	.	.	.	.	.	.	.	.	$-F$	F	.	.	.	.	.	.	.	.	.	.	.
$X.44$	$-/E$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−2	H	$/H$
$X.45$	$-E$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−2	$/H$	H
$X.46$	.	−3	.	.	.	.	.	3	.	−3	.	.	.	.	.	.	6	−3	.	.	.	.	.	.	.
$X.47$	.	3	.	.	.	.	.	−3	.	3	.	.	.	.	.	.	6	−3	.	.	.	.	.	.	.
$X.48$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	12	−6	.	.	.	.	.	.	.
$X.49$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−6	3	.	.	.	.	.	.	.
$X.50$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−6	3	.	.	.	.	.	.	.
$X.51$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−6	3	.	.	.	.	.	.	.
$X.52$	.	.	.	.	−3	3	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−3	.	.
$X.53$	.	.	.	.	3	.	−3	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−3	.	.
$X.54$	.	.	.	.	.	−3	3	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−3	.	.
$X.55$	.	−3	.	.	.	.	.	.	−3	3	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.56$	.	3	.	.	.	.	.	.	3	−3	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.57$	.	−3	.	.	.	.	.	−3	3	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.58$	.	3	.	.	.	.	.	3	−3	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.59$	.	.	.	.	−3	3	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.60$	.	.	.	.	.	−3	3	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.61$	.	.	.	.	3	.	−3	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.62$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	3	.	.

(*continued on next page*)

Table 10.9 (*continued*)

2	3	1	1	1	3	3	3	3	.	.	.	2	2	2
3	2	2	2	2	1	1	1	1	3	3	3	2	2	2
	$18a$	$18b$	$18c$	$18d$	$24a$	$24b$	$24c$	$24d$	$27a$	$27b$	$27c$	$36a$	$36b$	$36c$
$2P$	$9a$	$9f$	$9c$	$9e$	$12b$	$12c$	$12b$	$12c$	$27a$	$27c$	$27b$	$18a$	$18a$	$18a$
$3P$	$6a$	$6a$	$6d$	$6d$	$8a$	$8a$	$8b$	$8b$	$9b$	$9b$	$9b$	$12a$	$12a$	$12a$
$X.1$	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$X.2$	1	1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1
$X.3$	-2	1	.	.	F	F	$-F$	$-F$	-1	-1	-1	.	.	.
$X.4$	-2	1	.	.	$-F$	$-F$	F	F	-1	-1	-1	.	.	.
$X.5$	2	-1	.	.	.	.	.	.	-1	-1	-1	2	2	2
$X.6$	3	.	1	1	-1	-1	-1	-1	.	.	.	-1	-1	-1
$X.7$	3	.	-1	-1	1	1	1	1	.	.	.	-1	-1	-1
$X.8$	-4	-1	.	.	.	.	.	.	1	1	1	.	.	.
$X.9$	1	1	.	.	.	.	.	.	.	.	.	-1	-1	-1
$X.10$	1	-2	.	.	.	.	.	.	.	.	.	-1	-1	-1
$X.11$	1	1	.	.	.	.	.	.	.	.	.	-1	-1	-1
$X.12$	.	.	-1	-1	.	.	.	.	-1	-1	-1	.	.	.
$X.13$	.	.	1	1	.	.	.	.	-1	-1	-1	.	.	.
$X.14$	.	.	-1	2	.	.	.	.	2	-1	-1	.	.	.
$X.15$	.	.	1	-2	.	.	.	.	2	-1	-1	.	.	.
$X.16$	-2	1	.	.	.	.	.	.	.	.	.	.	.	.
$X.17$	-2	1	.	.	.	.	.	.	.	.	.	.	.	.
$X.18$	-2	-2	.	.	.	.	.	.	.	.	.	.	.	.
$X.19$	.	.	.	.	.	.	.	.	1	1	1	.	.	.
$X.20$	.	.	.	.	.	.	.	.	1	J	$/J$	.	.	.
$X.21$	.	.	.	.	.	.	.	.	1	$/J$	J	.	.	.
$X.22$	.	.	.	.	.	.	.	.	-2	1	1	.	.	.
$X.23$	.	.	.	.	.	.	.	.	-2	1	1	.	.	.
$X.24$	3	.	.	.	.	.	.	.	.	.	.	1	1	1
$X.25$	.	.	1	1	.	.	.	.	.	.	.	.	.	.
$X.26$	.	.	-1	-1	.	.	.	.	.	.	.	.	.	.
$X.27$	.	.	-2	1	.	.	.	.	.	.	.	.	.	.
$X.28$	.	.	2	-1	.	.	.	.	.	.	.	.	.	.
$X.29$	.	.	.	.	G	$/G$	G	$/G$	.	.	.	.	.	.
$X.30$	.	.	.	.	$-G$	$-/G$	$-G$	$-/G$	.	.	.	.	.	.
$X.31$	.	.	.	.	$/G$	G	$/G$	G	.	.	.	.	.	.
$X.32$	.	.	.	.	$-/G$	$-G$	$-/G$	$-G$	.	.	.	.	.	.
$X.33$	.	.	.	.	$/G$	G	$/G$	G	.	.	.	2	$-H$	$-/H$
$X.34$	.	.	.	.	$-/G$	$-G$	$-/G$	$-G$	.	.	.	2	$-H$	$-/H$
$X.35$	.	.	.	.	G	$/G$	G	$/G$	.	.	.	2	$-/H$	$-H$
$X.36$	.	.	.	.	$-G$	$-/G$	$-G$	$-/G$	.	.	.	2	$-/H$	$-H$
$X.37$	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.38$	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.39$	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.40$	.	.	.	.	I	$-/I$	$-I$	$/I$	.	.	.	.	.	.
$X.41$	.	.	.	.	$-I$	$/I$	I	$-/I$	.	.	.	.	.	.
$X.42$	.	.	.	.	$-/I$	I	$/I$	$-I$	.	.	.	.	.	.
$X.43$	.	.	.	.	$/I$	$-I$	$-/I$	I	.	.	.	.	.	.
$X.44$	.	.	.	.	.	.	.	.	.	.	.	-2	H	$/H$
$X.45$	.	.	.	.	.	.	.	.	.	.	.	-2	$/H$	H
$X.46$	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.47$	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.48$	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.49$	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.50$	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.51$	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.52$	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.53$	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.54$	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.55$	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.56$	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.57$	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.58$	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.59$	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.60$	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.61$	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$X.62$	.	.	.	.	.	.	.	.	.	.	.	.	.	.

Notes. Here $A = -\frac{1}{2}(27 - 27i\sqrt{3})$, $B = -27 - 27i\sqrt{3}$, $C = -\frac{1}{2}(3 - 9i\sqrt{3})$, $D = -\frac{1}{2}(3 - 3i\sqrt{3})$, $E = 3 + 3i\sqrt{3}$, $F = i\sqrt{2}$, $G = \frac{1}{2}(1 + \sqrt{3})$, $H = 1 + i\sqrt{3}$, $I = \xi^{11} + \xi^{17}$, $J = -\frac{1}{2}(1 + 3i\sqrt{3})$, where ξ is a 24th root of unity.

Table 10.10

Character table of $N_7 = N_G(s_1)$

2	4	4	2	4	4	1	1	1	3	4	4	2	2	1	1	1	3	.	.	.	3
3	2	1	2	2	2	2	2	2	1	1	1	2	2	2	2	2	1	1	.	.	1
7	2	1	.	1	1	1	.	.	1	.	.	.	.	.	.	.	2	2	2	2	.
	1a	2a	2b	3a	3b	3c	3d	3e	4a	6a	6b	6c	6d	6e	6f	6g	7a	7b	7c	7d	8a
2P	1a	1a	1a	3b	3a	3c	3e	3d	2a	3b	3a	3b	3a	3c	3e	3d	7a	7b	7c	7d	4a
3P	1a	2a	2b	1a	1a	1a	1a	1a	4a	2a	2a	2b	2b	2b	2b	2b	7a	7b	7c	7d	8b
5P	1a	2a	2b	3b	3a	3c	3e	3d	4a	6b	6a	6d	6c	6e	6g	6f	7a	7b	7c	7d	8b
7P	1a	2a	2b	3a	3b	3c	3d	3e	4a	6a	6b	6c	6d	6e	6f	6g	1a	1a	1a	1a	8a
11P	1a	2a	2b	3b	3a	3c	3e	3d	4a	6b	6a	6d	6c	6e	6g	6f	7a	7b	7c	7d	8b
13P	1a	2a	2b	3a	3b	3c	3d	3e	4a	6a	6b	6c	6d	6e	6f	6g	7a	7b	7c	7d	8b
17P	1a	2a	2b	3b	3a	3c	3e	3d	4a	6b	6a	6d	6c	6e	6g	6f	7a	7b	7c	7d	8a
19P	1a	2a	2b	3a	3b	3c	3d	3e	4a	6a	6b	6c	6d	6e	6f	6g	7a	7b	7c	7d	8b
23P	1a	2a	2b	3b	3a	3c	3e	3d	4a	6b	6a	6d	6c	6e	6g	6f	7a	7b	7c	7d	8a
X.1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.2	1	1	−1	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	1	1	1	1	−1
X.3	1	1	1	A	/A	1	A	/A	1	A	/A	A	/A	1	A	/A	1	1	1	1	1
X.4	1	1	−1	A	/A	1	A	/A	1	A	/A	−A	−/A	−1	−A	−/A	1	1	1	1	−1
X.5	1	1	1	/A	A	1	/A	A	1	/A	A	/A	A	1	/A	A	1	1	1	1	1
X.6	1	1	−1	/A	A	1	/A	A	1	/A	A	−/A	−A	−1	−/A	−A	1	1	1	1	−1
X.7	6	6	.	.	.	6	.	.	6	.	.	.	.	.	.	.	−1	6	−1	−1	.
X.8	6	2	.	B	/B	.	.	.	.	E	/E	.	.	.	.	.	6	−1	−1	−1	F
X.9	6	2	.	B	/B	.	.	.	.	E	/E	.	.	.	.	.	6	−1	−1	−1	−F
X.10	6	−2	.	B	/B	.	.	.	2	−E	−/E	.	.	.	.	.	6	−1	−1	−1	.
X.11	6	2	.	6	6	.	.	.	.	2	2	.	.	.	.	.	6	−1	−1	−1	F
X.12	6	2	.	6	6	.	.	.	.	2	2	.	.	.	.	.	6	−1	−1	−1	−F
X.13	6	−2	.	6	6	.	.	.	2	−2	−2	.	.	.	.	.	6	−1	−1	−1	.
X.14	6	2	.	/B	B	.	.	.	.	/E	E	.	.	.	.	.	6	−1	−1	−1	F
X.15	6	2	.	/B	B	.	.	.	.	/E	E	.	.	.	.	.	6	−1	−1	−1	−F
X.16	6	−2	.	/B	B	.	.	.	2	−/E	−E	.	.	.	.	.	6	−1	−1	−1	.
X.17	7	−1	1	C	/C	1	A	/A	−1	−A	−/A	A	/A	1	A	/A	7	.	.	.	−1
X.18	7	−1	−1	C	/C	1	A	/A	−1	−A	−/A	−A	−/A	−1	−A	−/A	7	.	.	.	1
X.19	7	−1	1	7	7	1	1	1	−1	−1	−1	1	1	1	1	1	7	.	.	.	−1
X.20	7	−1	−1	7	7	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1	7	.	.	.	1
X.21	7	−1	1	/C	C	1	/A	A	−1	−/A	−A	/A	A	1	/A	A	7	.	.	.	−1
X.22	7	−1	−1	/C	C	1	/A	A	−1	−/A	−A	−/A	−A	−1	−/A	−A	7	.	.	.	1
X.23	8	.	2	D	/D	−1	−A	−/A	.	.	.	E	/E	−1	−A	−/A	8	1	1	1	.
X.24	8	.	−2	D	/D	−1	−A	−/A	.	.	.	−E	−/E	1	A	/A	8	1	1	1	.
X.25	8	.	2	8	8	−1	−1	−1	.	.	.	2	2	−1	−1	−1	8	1	1	1	.
X.26	8	.	−2	8	8	−1	−1	−1	.	.	.	−2	−2	1	1	1	8	1	1	1	.
X.27	8	.	2	/D	D	−1	−/A	−A	.	.	.	/E	E	−1	−/A	−A	8	1	1	1	.
X.28	8	.	−2	/D	D	−1	−/A	−A	.	.	.	−/E	−E	1	/A	A	8	1	1	1	.
X.29	18	−6	.	.	.	.	.	.	6	.	.	.	.	.	.	.	−3	−3	−3	4	.
X.30	18	−6	.	.	.	.	.	.	6	.	.	.	.	.	.	.	−3	−3	4	−3	.
X.31	36	12	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−6	−6	1	1	.
X.32	42	−6	.	.	.	6	.	.	−6	.	.	.	.	.	.	.	−7	.	.	.	.
X.33	48	.	.	.	.	−6	.	.	.	.	.	.	.	.	.	.	−8	6	−1	−1	.

2	3	3	3	3	.	.	.	3	3	3	3	2
3	1	1	1	.	1	1	1	1	1	1	1	.
7	.	.	.	1	1	1	1	.	.	.	.	1
	8b	12a	12b	14a	21a	21b	21c	24a	24b	24c	24d	28a
2P	4a	6a	6b	7a	21a	21c	21b	12a	12b	12a	12b	14a
3P	8a	4a	4a	14a	7a	7b	7b	8b	8a	8a	8b	28a
5P	8a	12b	12a	14a	21a	21c	21b	24b	24a	24d	24c	28a
7P	8b	12a	12b	2a	3c	3a	3b	24a	24b	24c	24d	4a
11P	8a	12b	12a	14a	21a	21c	21b	24b	24a	24d	24c	28a
13P	8a	12a	12b	14a	21a	21b	21c	24c	24d	24a	24b	28a
17P	8b	12b	12a	14a	21a	21c	21b	24d	24c	24b	24a	28a
19P	8a	12a	12b	14a	21a	21b	21c	24c	24d	24a	24b	28a
23P	8b	12b	12a	14a	21a	21c	21b	24d	24c	24b	24a	28a
X.1	1	1	1	1	1	1	1	1	1	1	1	1
X.2	−1	1	1	1	1	1	1	−1	−1	−1	−1	1
X.3	1	/A	A	1	1	A	/A	A	/A	A	/A	1
X.4	−1	/A	A	1	1	A	/A	−A	−/A	−A	−/A	1
X.5	1	A	/A	1	1	/A	A	/A	A	/A	A	1
X.6	−1	A	/A	1	1	/A	A	−/A	−A	−/A	−A	1
X.7	.	.	.	−1	−1	.	.	.	.	.	.	−1
X.8	−F	.	.	2	.	−A	−/A	G	−/G	−G	/G	.
X.9	F	.	.	2	.	−A	−/A	−G	/G	G	−/G	.
X.10	.	/E	E	−2	.	−A	−/A	.	.	.	.	2
X.11	−F	.	.	2	.	−1	−1	F	−F	−F	F	.
X.12	F	.	.	2	.	−1	−1	−F	F	F	−F	.
X.13	.	2	2	−2	.	−1	−1	.	.	.	.	2
X.14	−F	.	.	2	.	−/A	−A	/G	−G	−/G	G	.
X.15	F	.	.	2	.	−/A	−A	−/G	G	/G	−G	.
X.16	.	E	/E	−2	.	−/A	−A	.	.	.	.	2
X.17	−1	−/A	−A	−1	1	.	.	−A	−/A	−A	−/A	−1
X.18	1	−/A	−A	−1	1	.	.	A	/A	A	/A	−1
X.19	−1	−1	−1	−1	1	.	.	−1	−1	−1	−1	−1
X.20	1	−1	−1	−1	1	.	.	1	1	1	1	−1
X.21	−1	−A	−/A	−1	1	.	.	−/A	−A	−/A	−A	−1
X.22	1	−A	−/A	−1	1	.	.	/A	A	/A	A	−1
X.23	.	.	.	.	−1	A	/A	.	.	.	.	.
X.24	.	.	.	.	−1	A	/A	.	.	.	.	.
X.25	.	.	.	.	−1	1	1	.	.	.	.	.
X.26	.	.	.	.	−1	1	1	.	.	.	.	.
X.27	.	.	.	.	−1	/A	A	.	.	.	.	.
X.28	.	.	.	.	−1	/A	A	.	.	.	.	.
X.29	.	.	.	1	.	.	.	.	.	.	.	−1
X.30	.	.	.	1	.	.	.	.	.	.	.	−1
X.31	.	.	.	−2	.	.	.	.	.	.	.	.
X.32	.	.	.	1	−1	.	.	.	.	.	.	1
X.33	.	.	.	.	1	.	.	.	.	.	.	.

Notes. Here $A = -\frac{1}{2}(1 - i\sqrt{3})$, $B = -3(1 - i\sqrt{3})$, $C = -\frac{7}{2}(1 - i\sqrt{3})$, $D = -4(1 - i\sqrt{3})$, $E = -1 + i\sqrt{3}$, $F = \sqrt{2}$ and $G = \xi - \xi^{19}$, where ξ is a 24th root of unity.

Table 10.11
Character table of $N_5 = N_G(f)$

2	5	5	4	2	5	5	4	4	4	4	4	3	2	2	3	3	3	2	2	2
3	1	1	.	1	1	1	.	.	.	.	.	1	.	1	.	.	1	.	1	1
5	3	1	1	1	.	.	1	1	1	.	.	3	2	1	.	.	1	1	.	.
	1a	2a	2b	3a	4a	4b	4c	4d	4e	4f	4g	5a	5b	6a	8a	8b	10a	10b	12a	12b
2P	1a	1a	1a	3a	2a	2a	2a	2b	2b	2b	2b	5a	5b	3a	4a	4b	5a	5b	6a	6a
3P	1a	2a	2b	1a	4b	4a	4c	4e	4d	4g	4f	5a	5b	2a	8b	8a	10a	10b	4b	4a
5P	1a	2a	2b	3a	4a	4b	4c	4d	4e	4f	4g	1a	1a	6a	8a	8b	2a	2b	12a	12b
7P	1a	2a	2b	3a	4b	4a	4c	4e	4d	4g	4f	5a	5b	6a	8b	8a	10a	10b	12b	12a
11P	1a	2a	2b	3a	4b	4a	4c	4e	4d	4g	4f	5a	5b	6a	8b	8a	10a	10b	12b	12a
13P	1a	2a	2b	3a	4a	4b	4c	4d	4e	4f	4g	5a	5b	6a	8a	8b	10a	10b	12a	12b
17P	1a	2a	2b	3a	4a	4b	4c	4d	4e	4f	4g	5a	5b	6a	8a	8b	10a	10b	12a	12b
19P	1a	2a	2b	3a	4b	4a	4c	4e	4d	4g	4f	5a	5b	6a	8b	8a	10a	10b	12b	12a
23P	1a	2a	2b	3a	4b	4a	4c	4e	4d	4g	4f	5a	5b	6a	8b	8a	10a	10b	12b	12a
29P	1a	2a	2b	3a	4a	4b	4c	4d	4e	4f	4g	5a	5b	6a	8a	8b	10a	10b	12a	12b
X.1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.2	1	1	−1	1	−1	−1	1	C	−C	C	−C	1	1	1	C	−C	1	−1	−1	−1
X.3	1	1	1	1	1	1	1	−1	−1	−1	−1	1	1	1	−1	−1	1	1	1	1
X.4	1	1	−1	1	−1	−1	1	−C	C	−C	C	1	1	1	−C	C	1	−1	−1	−1
X.5	2	2	2	−1	2	2	2	.	.	.	.	2	2	−1	.	.	2	2	−1	−1
X.6	2	2	−2	−1	−2	−2	2	.	.	.	.	2	2	−1	.	.	2	−2	1	1
X.7	2	−2	.	−1	A	−A	.	D	/D	−D	−/D	2	2	1	.	.	−2	.	−C	C
X.8	2	−2	.	−1	−A	A	.	/D	D	−/D	−D	2	2	1	.	.	−2	.	C	−C
X.9	2	−2	.	−1	−A	A	.	−/D	−D	/D	D	2	2	1	.	.	−2	.	C	−C
X.10	2	−2	.	−1	A	−A	.	−D	−/D	D	/D	2	2	1	.	.	−2	.	−C	C
X.11	3	3	−1	.	3	3	−1	1	1	1	1	3	3	.	−1	−1	3	−1	.	.
X.12	3	3	1	.	−3	−3	−1	−C	C	−C	C	3	3	.	C	−C	3	1	.	.
X.13	3	3	1	.	−3	−3	−1	C	−C	C	−C	3	3	.	−C	C	3	1	.	.
X.14	3	3	−1	.	3	3	−1	−1	−1	−1	−1	3	3	.	1	1	3	−1	.	.
X.15	4	−4	.	1	B	−B	.	.	.	.	.	4	4	−1	.	.	−4	.	−C	C
X.16	4	−4	.	1	−B	B	.	.	.	.	.	4	4	−1	.	.	−4	.	C	−C
X.17	20	4	.	2	.	.	−4	.	.	.	.	−5	.	−2	.	.	−1	.	.	.
X.18	20	4	.	2	.	.	−4	.	.	.	.	−5	.	−2	.	.	−1	.	.	.
X.19	20	4	.	−4	.	.	−4	.	.	.	.	−5	.	4	.	.	−1	.	.	.
X.20	24	.	4	.	.	.	.	4	4	.	.	24	−1	.	.	.	.	−1	.	.
X.21	24	.	−4	.	.	.	.	−B	B	.	.	24	−1	.	.	.	.	1	.	.
X.22	24	.	−4	.	.	.	.	B	−B	.	.	24	−1	.	.	.	.	1	.	.
X.23	24	.	4	.	.	.	.	−4	−4	.	.	24	−1	.	.	.	.	−1	.	.
X.24	40	−8	.	4	.	.	.	.	.	.	.	−10	.	4	.	.	2	.	.	.
X.25	40	−8	.	−2	.	.	.	.	.	.	.	−10	.	−2	.	.	2	.	.	.
X.26	40	−8	.	−2	.	.	.	.	.	.	.	−10	.	−2	.	.	2	.	.	.
X.27	60	12	.	.	.	.	4	.	.	.	.	−15	.	.	.	.	−3	.	.	.

2	1	1	2	2	2	1	1
3	1	1	.	.	.	1	1
5	1	1	1	1	1	1	1
	15a	15b	20a	20b	20c	30a	30b
2P	15a	15b	10a	10b	10b	15a	15b
3P	5a	5a	20a	20c	20b	10a	10a
5P	3a	3a	4c	4d	4e	6a	6a
7P	15b	15a	20a	20c	20b	30b	30a
11P	15b	15a	20a	20c	20b	30b	30a
13P	15b	15a	20a	20b	20c	30b	30a
17P	15a	15b	20a	20b	20c	30a	30b
19P	15a	15b	20a	20c	20b	30a	30b
23P	15a	15b	20a	20c	20b	30a	30b
29P	15b	15a	20a	20b	20c	30b	30a
X.1	1	1	1	1	1	1	1
X.2	1	1	1	C	−C	1	1
X.3	1	1	1	−1	−1	1	1
X.4	1	1	1	−C	C	1	1
X.5	−1	−1	2	.	.	−1	−1
X.6	−1	−1	2	.	.	−1	−1
X.7	−1	−1	.	D	/D	1	1
X.8	−1	−1	.	/D	D	1	1
X.9	−1	−1	.	−/D	−D	1	1
X.10	−1	−1	.	−D	−/D	1	1
X.11	.	.	−1	1	1	.	.
X.12	.	.	−1	−C	C	.	.
X.13	.	.	−1	C	−C	.	.
X.14	.	.	−1	−1	−1	.	.
X.15	1	1	.	.	.	−1	−1
X.16	1	1	.	.	.	−1	−1
X.17	E	/E	1	.	.	−E	−/E
X.18	/E	E	1	.	.	−/E	−E
X.19	1	1	1	.	.	−1	−1
X.20	.	.	.	−1	−1	.	.
X.21	.	.	.	−C	C	.	.
X.22	.	.	.	C	−C	.	.
X.23	.	.	.	1	1	.	.
X.24	−1	−1	.	.	.	−1	−1
X.25	−/E	−E	.	.	.	−/E	−E
X.26	−E	−/E	.	.	.	−E	−/E
X.27	.	.	−1	.	.	.	.

Notes. Here $A = 2i$, $B = -4i$, $C = i$, $D = 1 - i$ and $E = -\frac{1}{2}(1 + i\sqrt{15})$.

11. Partial character table of matrix group $\mathfrak{G}$

11.1. Intersection matrix M_2

$$D_2 = \begin{pmatrix}
. & 1 & . & . & . & . & . & . & . & . & . \\
17199 & 78 & 180 & . & 27 & 18 & . & . & . & 9 & 1 \\
. & 480 & 27 & . & . & 18 & 21 & 39 & . & . & 3 \\
. & . & . & 147 & . & 48 & . & . & 48 & . & 24 \\
. & 2048 & . & . & 81 & 144 & 210 & 390 & 72 & 90 & 152 \\
. & 2304 & 864 & 588 & 243 & 363 & 189 & 351 & 156 & 297 & 264 \\
. & . & 2304 & . & 810 & 432 & 189 & 234 & 648 & 1170 & 588 \\
. & . & 6912 & . & 2430 & 1296 & 378 & 585 & 648 & 1890 & 936 \\
. & . & . & 2352 & 486 & 624 & 1134 & 702 & 1587 & 918 & 1068 \\
. & 6144 & . & . & 810 & 1584 & 2730 & 2730 & 1224 & 549 & 1364 \\
. & 6144 & 6912 & 14112 & 12312 & 12672 & 12348 & 12168 & 12816 & 12276 & 12799
\end{pmatrix}$$

11.2. Intersection matrix M_3

$$D_3 = \begin{pmatrix}
. & . & 1 & . & . & . & . & . & . & . & . \\
. & 480 & 27 & . & . & 18 & 21 & 39 & . & . & 3 \\
45864 & 72 & 684 & . & 99 & 18 & . & . & . & 81 & 9 \\
. & . & . & 588 & . & 48 & . & . & 48 & . & 72 \\
. & . & 2816 & . & 1134 & 720 & 126 & 234 & 216 & 774 & 408 \\
. & 2304 & 864 & 588 & 1215 & 1380 & 441 & 819 & 444 & 621 & 720 \\
. & 6144 & . & . & 486 & 1008 & 3570 & 2886 & 1944 & 702 & 1524 \\
. & 18432 & . & . & 1458 & 3024 & 4662 & 5382 & 1944 & 1134 & 2520 \\
. & . & . & 2352 & 1458 & 1776 & 3402 & 2106 & 4332 & 2754 & 2772 \\
. & . & 20736 & . & 6966 & 3312 & 1638 & 1638 & 3672 & 7290 & 3612 \\
. & 18432 & 20736 & 42336 & 33048 & 34560 & 32004 & 32760 & 33264 & 32508 & 34224
\end{pmatrix}$$

11.3. Intersection matrix M_5

$$D_5 = \begin{pmatrix}
. & . & . & . & 1 & . & . & . & . & . & . \\
. & 2048 & . & . & 81 & 144 & 210 & 390 & 72 & 90 & 152 \\
. & . & 2816 & . & 1134 & 720 & 126 & 234 & 216 & 774 & 408 \\
. & . & . & 1568 & 972 & 1344 & 1764 & 1404 & 2064 & 1620 & 1632 \\
1304576 & 6144 & 32256 & 7056 & 15784 & 13248 & 10584 & 10296 & 10944 & 14616 & 11776 \\
. & 18432 & 34560 & 16464 & 22356 & 20000 & 21420 & 21060 & 19920 & 20844 & 19824 \\
. & 61440 & 13824 & 49392 & 40824 & 48960 & 52808 & 54600 & 44352 & 36072 & 46080 \\
. & 184320 & 41472 & 63504 & 64152 & 77760 & 88200 & 92456 & 69120 & 62856 & 73728 \\
. & 36864 & 41472 & 101136 & 73872 & 79680 & 77616 & 74880 & 82496 & 76464 & 81120 \\
. & 61440 & 198144 & 105840 & 131544 & 111168 & 84168 & 90792 & 101952 & 128168 & 107008 \\
. & 933888 & 940032 & 959616 & 953856 & 951552 & 967680 & 958464 & 973440 & 963072 & 962848
\end{pmatrix}$$

11.4. Common eigenvectors of M_i

$$\begin{array}{ccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \frac{-1}{7} & \frac{1}{7} & 0 & \frac{1}{28} & 0 & \frac{-1}{28} & \frac{-1}{28} & 0 & \frac{1}{28} & 0 \\
1 & \frac{17}{637} & \frac{5}{98} & \frac{-9}{208} & \frac{-5}{832} & \frac{-81}{5096} & \frac{89}{5824} & \frac{5}{3136} & \frac{9}{784} & \frac{313}{40768} & \frac{-45}{20384} \\
1 & \frac{1}{13} & \frac{11}{182} & \frac{-5}{208} & \frac{73}{5824} & \frac{5}{728} & \frac{1}{832} & \frac{1}{112} & \frac{-25}{2912} & \frac{1}{208} & \frac{-5}{5824} \\
1 & \frac{-1}{39} & \frac{1}{91} & \frac{-1}{624} & \frac{19}{5824} & \frac{1}{546} & \frac{7}{4992} & \frac{-1}{1344} & \frac{1}{8736} & \frac{-1}{1664} & \frac{-1}{34944} \\
1 & \frac{-1}{147} & \frac{17}{1274} & \frac{5}{624} & \frac{-29}{40768} & \frac{-5}{2184} & \frac{11}{17472} & \frac{-1}{9408} & \frac{-25}{30576} & \frac{1}{40768} & \frac{5}{61152} \\
1 & \frac{31}{1323} & \frac{41}{3822} & \frac{5}{936} & \frac{1}{11648} & \frac{55}{45864} & \frac{-29}{104832} & \frac{-1}{56448} & \frac{125}{183456} & \frac{-127}{733824} & \frac{-55}{825552} \\
1 & \frac{11}{1911} & \frac{-1}{1274} & \frac{-1}{624} & \frac{-139}{81536} & \frac{1}{1176} & \frac{19}{34944} & \frac{-3}{6272} & \frac{-23}{61152} & \frac{11}{81536} & \frac{1}{30576} \\
1 & \frac{-1}{147} & \frac{5}{1274} & \frac{-1}{624} & \frac{-29}{40768} & \frac{1}{15288} & \frac{-1}{2496} & \frac{1}{4704} & \frac{1}{8736} & \frac{-1}{20384} & \frac{1}{122304} \\
1 & \frac{11}{1911} & \frac{-1}{1274} & \frac{-1}{624} & \frac{25}{40768} & \frac{-1}{1911} & \frac{-1}{17472} & \frac{-1}{9408} & \frac{-1}{30576} & \frac{-5}{40768} & \frac{1}{30576} \\
1 & \frac{-5}{1911} & \frac{-5}{1274} & \frac{1}{624} & \frac{1}{5824} & \frac{1}{7644} & \frac{1}{17472} & \frac{1}{9408} & \frac{1}{30576} & \frac{5}{40768} & \frac{-1}{30576}
\end{array}$$

Table 11.5
Gollan–Ostermann numbers of $\mathfrak{G}$

	r_1	r_2	r_3	r_4	r_5	r_6
$1a$	143127000	0	0	0	0	0
$2a$	10200	483840	1935360	0	6881280	0
$3a$	3510	132678	2388204	0	5031936	1061424
$3b$	243	59049	78732	0	1271376	2519424
$3c$	540	14580	145800	0	1831680	2332800
$4a$	120	2016	653184	0	3526656	1548288
$4b$	40	40560	14400	230400	1064960	2465280
$5a$	0	40500	40500	144000	1260000	2358000
$6a$	60	10260	154440	103680	1896960	2177280
$6b$	54	31302	89964	96768	1317888	2063664
$6c$	3	21033	42876	165888	1271184	2135808
$7a$	9	14455	121128	117600	1645616	2126208
$8a$	8	12656	76512	171264	1435136	2106240
$8b$	0	19368	37872	187776	1266048	2228544
$9a$	0	29160	15552	116640	800928	1621296
$9b$	0	11664	44712	186624	1150848	1831248
$9c$	9	11907	81324	155520	1507968	2153952
$10a$	0	12660	63540	167040	1446240	2231280
$12a$	6	16854	51660	164736	1344768	2253168
$12b$	6	16854	51660	164736	1344768	2253168
$12c$	3	19089	51732	169344	1348080	2234304
$12d$	4	15228	55512	172800	1342976	2160000
$13a$	3	12805	59280	180960	1398800	2203968
$14a$	1	23247	46872	163296	1281392	2189376
$15a$	0	15300	51300	178560	1347840	2202480
$15b$	0	15300	51300	178560	1347840	2202480
$18a$	0	13608	41472	171936	1243296	2106864
$18b$	3	17577	51084	164160	1292352	2136672
$19a$	0	16796	45372	178752	1304464	2198832
$20a$	0	16580	46020	180480	1305760	2191920
$21a$	3	14245	51072	188160	1360352	2246160
$24a$	2	16778	45948	179904	1301312	2195664
$24b$	2	16778	45948	179904	1301312	2195664
$24c$	0	18936	37440	189504	1264320	2232864
$24d$	0	18936	37440	189504	1264320	2232864
$27a$	0	22032	34344	168480	1174176	2112480
$27b$	0	13284	46980	191808	1337472	2217456
$27c$	0	13284	46980	191808	1337472	2217456
$28a$	1	19607	38640	188832	1273104	2242464
$30a$	0	22860	43740	169920	1287360	2232720
$30b$	0	22860	43740	169920	1287360	2232720
$31a$	0	16740	43524	187488	1312416	2236464
$31b$	0	16740	43524	187488	1312416	2236464
$36a$	0	15768	42768	182304	1295136	2194128
$36b$	0	15768	42768	182304	1295136	2194128
$36c$	0	15768	42768	182304	1295136	2194128
$39a$	0	20176	39624	180960	1259024	2203968
$39b$	0	20176	39624	180960	1259024	2203968

(*continued on next page*)

Table 11.5 (*continued*)

	r_7	r_8	r_9	r_{10}	r_{11}
$1a$	0	0	0	0	0
$2a$	17694720	0	23224320	30965760	61931520
$3a$	4852224	2286144	4245696	55194048	67931136
$3b$	8817984	11022480	8188128	10707552	100462032
$3c$	5365440	9331200	7931520	13296960	102876480
$4a$	2985984	4644864	7064064	27740160	94961664
$4b$	5575680	9400320	8352000	9154560	106828800
$5a$	5328000	9180000	8460000	11304000	105012000
$6a$	3931200	6428160	8346240	16632000	103446720
$6b$	5736960	8589888	9153216	12772800	103274496
$6c$	5164992	8176464	9012384	11982816	105153552
$7a$	4172448	6597360	8852928	15501248	103978000
$8a$	4697088	7335936	9179904	13258240	104854016
$8b$	5136768	8401536	8696448	11226240	105926400
$9a$	7709904	9961056	10672560	9148464	103051440
$9b$	5155488	7441632	9762768	12286080	105255936
$9c$	4475088	7138368	8852976	13715568	105034320
$10a$	4691520	7693920	8655840	12679680	105485280
$12a$	5078016	8244288	8640576	11838144	105494784
$12b$	5078016	8244288	8640576	11838144	105494784
$12c$	5035392	8282736	8674560	11839392	105472368
$12d$	4931712	7848576	8899200	12248064	105452928
$13a$	4673136	7546032	8810880	12578384	105662752
$14a$	5290272	8585136	8757504	11527488	105262416
$15a$	4842000	7845120	8812080	12167280	105665040
$15b$	4842000	7845120	8812080	12167280	105665040
$18a$	5159376	7939296	9260784	11743920	105446448
$18b$	5193072	8149248	9009360	11915856	105197616
$19a$	5033328	8128656	8799888	11743216	105677696
$20a$	5033760	8131680	8796000	11741920	105682880
$21a$	4784016	7816032	8708784	12064304	105893872
$24a$	5039232	8123904	8789376	11745088	105689792
$24b$	5039232	8123904	8789376	11745088	105689792
$24c$	5138496	8406720	8698176	11234880	105905664
$24d$	5138496	8406720	8698176	11234880	105905664
$27a$	5530032	8670240	9083664	11032848	105298704
$27b$	4760208	7760448	8780400	12020400	105998544
$27c$	4760208	7760448	8780400	12020400	105998544
$28a$	5135712	8401680	8720544	11216128	105890288
$30a$	5213520	8570880	8660880	11441520	105483600
$30b$	5213520	8570880	8660880	11441520	105483600
$31a$	4955040	8115552	8704800	11651040	105903936
$31b$	4955040	8115552	8704800	11651040	105903936
$36a$	5031504	8048160	8888400	11676528	105752304
$36b$	5031504	8048160	8888400	11676528	105752304
$36c$	5031504	8048160	8888400	11676528	105752304
$39a$	5212272	8416944	8810880	11320400	105662752
$39b$	5212272	8416944	8810880	11320400	105662752

Table 11.6
11 characters of 𝔊

2	15	15	6	3	4	11	9	3	4	6	3	3	7	5	3	.	1
3	10	4	7	10	7	3	1	1	3	3	4	1	1	1	6	6	4
5	3	1	.	.	1	.	1	3	1	.	.	.	.	.	.	.	.
7	2	1	1	.	.	1	.	.	.	.	.	2	.	.	.	.	.
13	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
19	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
31	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
	1a	2a	3a	3b	3c	4a	4b	5a	6a	6b	6c	7a	8a	8b	9a	9b	9c
1P	1a	2a	3a	3b	3c	4a	4b	5a	6a	6b	6c	7a	8a	8b	9a	9b	9c
2P	1a	1a	3a	3b	3c	2a	2a	5a	3c	3a	3b	7a	4a	4b	9a	9b	9c
3P	1a	2a	1a	1a	1a	4a	4b	5a	2a	2a	2a	7a	8a	8b	3b	3b	3b
5P	1a	2a	3a	3b	3c	4a	4b	1a	6a	6b	6c	7a	8a	8b	9a	9b	9c
7P	1a	2a	3a	3b	3c	4a	4b	5a	6a	6b	6c	1a	8a	8b	9a	9b	9c
11P	1a	2a	3a	3b	3c	4a	4b	5a	6a	6b	6c	7a	8a	8b	9a	9b	9c
13P	1a	2a	3a	3b	3c	4a	4b	5a	6a	6b	6c	7a	8a	8b	9a	9b	9c
17P	1a	2a	3a	3b	3c	4a	4b	5a	6a	6b	6c	7a	8a	8b	9a	9b	9c
19P	1a	2a	3a	3b	3c	4a	4b	5a	6a	6b	6c	7a	8a	8b	9a	9b	9c
23P	1a	2a	3a	3b	3c	4a	4b	5a	6a	6b	6c	7a	8a	8b	9a	9b	9c
29P	1a	2a	3a	3b	3c	4a	4b	5a	6a	6b	6c	7a	8a	8b	9a	9b	9c
31P	1a	2a	3a	3b	3c	4a	4b	5a	6a	6b	6c	7a	8a	8b	9a	9b	9c
37P	1a	2a	3a	3b	3c	4a	4b	5a	6a	6b	6c	7a	8a	8b	9a	9b	9c
X.1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.2	4123	27	64	−8	1	27	−5	−2	9	.	.	7	3	−1	−8	1	4
X.3	30875	155	104	14	5	27	−5	.	5	8	2	5	3	−1	14	5	2
X.4	61256	72	182	20	20	56	.	6	12	6	.	6	.	.	−7	−7	2
X.5	2450240	832	260	71	44	64	.	−10	4	4	−5	−5	.	.	17	−10	−1
X.6	3376737	609	819	9	9	161	1	−13	9	3	−3	.	1	1	9	9	.
X.7	4881384	1512	729	.	.	72	24	9	.	9	.	4	8	.	.	.	.
X.8	11577384	552	351	135	.	−120	24	9	.	15	3	7	−8	.	.	.	.
X.9	28861000	840	1078	−110	160	56	.	.	.	6	−6	.	.	.	−29	−2	−2
X.10	40199250	3410	−78	3	165	−62	10	.	5	2	−1	−7	−6	2	3	3	3
X.11	51684750	2190	.	108	135	−162	−10	.	15	.	12	−9	6	−2	.	.	.

2	3	5	5	2	3	.	3	1	1	3	1	.	2	.	3	3	3
3	1	2	2	3	1	1	.	1	1	2	2	.	.	1	1	1	1
5	1	.	.	.	.	.	.	1	1	.	.	.	1	.	.	.	.
7	.	.	.	.	.	.	1	.	.	.	.	.	.	1	.	.	.
13	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
19	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.
31	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
	10a	12a	12b	12c	12d	13a	14a	15a	15b	18a	18b	19a	20a	21a	24a	24b	24c
1P	10a	12a	12b	12c	12d	13a	14a	15a	15b	18a	18b	19a	20a	21a	24a	24b	24c
2P	5a	6b	6b	6c	6a	13a	7a	15a	15b	9a	9c	19a	10a	21a	12a	12b	12d
3P	10a	4a	4a	4a	4b	13a	14a	5a	5a	6c	6c	19a	20a	7a	8a	8a	8b
5P	2a	12b	12a	12c	12d	13a	14a	3c	3c	18a	18b	19a	4b	21a	24b	24a	24c
7P	10a	12a	12b	12c	12d	13a	2a	15b	15a	18a	18b	19a	20a	3a	24a	24b	24c
11P	10a	12b	12a	12c	12d	13a	14a	15b	15a	18a	18b	19a	20a	21a	24b	24a	24c
13P	10a	12a	12b	12c	12d	1a	14a	15b	15a	18a	18b	19a	20a	21a	24a	24b	24d
17P	10a	12b	12a	12c	12d	13a	14a	15a	15b	18a	18b	19a	20a	21a	24b	24a	24d
19P	10a	12a	12b	12c	12d	13a	14a	15a	15b	18a	18b	1a	20a	21a	24a	24b	24d
23P	10a	12b	12a	12c	12d	13a	14a	15a	15b	18a	18b	19a	20a	21a	24b	24a	24d
29P	10a	12b	12a	12c	12d	13a	14a	15b	15a	18a	18b	19a	20a	21a	24b	24a	24c
31P	10a	12a	12b	12c	12d	13a	14a	15a	15b	18a	18b	19a	20a	21a	24a	24b	24c
37P	10a	12a	12b	12c	12d	13a	14a	15b	15a	18a	18b	19a	20a	21a	24a	24b	24d
X.1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.2	2	.	.	.	1	2	−1	1	1	.	.	.	.	1	.	.	−1
X.3	.	.	.	.	1	.	1	.	.	2	2	.	.	−1	.	.	−1
X.4	2	2	2	2	.	.	2	.	.	−3	.	.	.	.	.	.	.
X.5	2	4	4	1	.	.	−1	−1	−1	1	1	.	.	1	.	.	.
X.6	−1	−1	−1	−1	1	.	.	−1	−1	−3	.	.	1	.	1	1	1
X.7	−3	−3	−3	.	.	1	.	.	.	.	.	−1	−1	1	−1	−1	.
X.8	−3	3	3	−3	.	.	−1	.	.	.	.	.	−1	1	1	1	.
X.9	.	2	2	2	.	−1	.	.	.	3	.	.	.	.	.	.	.
X.10	.	−2	−2	1	1	.	1	.	.	−1	−1	.	.	−1	.	.	−1
X.11	.	.	.	.	−1	.	−1	.	.	.	.	.	.	.	.	.	1

2	3	.	.	.	2	1	1	.	.	2	2	2	.	.
3	1	3	3	3	.	1	1	.	.	2	2	2	1	1
5	.	.	.	.	.	1	1	.	.	.	.	.	.	.
7	.	.	.	.	1	.	.	.	.	.	.	.	.	.
13	.	.	.	.	.	.	.	.	.	.	.	.	1	1
19	.	.	.	.	.	.	.	.	.	.	.	.	.	.
31	.	.	.	.	.	.	.	1	1	.	.	.	.	.
	24d	27a	27b	27c	28a	30a	30b	31a	31b	36a	36b	36c	39a	39b
1P	24d	27a	27b	27c	28a	30a	30b	31a	31b	36a	36b	36c	39a	39b
2P	12d	27a	27c	27b	14a	15a	15b	31a	31b	18a	18a	18a	39a	39b
3P	8b	9b	9b	9b	28a	10a	10a	31b	31a	12c	12c	12c	13a	13a
5P	24d	27a	27c	27b	28a	6a	6a	31a	31b	36b	36a	36c	39a	39b
7P	24d	27a	27b	27c	4a	30b	30a	31a	31b	36a	36b	36c	39b	39a
11P	24d	27a	27c	27b	28a	30b	30a	31b	31a	36b	36a	36c	39a	39b
13P	24c	27a	27b	27c	28a	30b	30a	31b	31a	36a	36b	36c	3a	3a
17P	24c	27a	27c	27b	28a	30a	30b	31b	31a	36b	36a	36c	39b	39a
19P	24c	27a	27b	27c	28a	30a	30b	31a	31b	36a	36b	36c	39b	39a
23P	24c	27a	27c	27b	28a	30a	30b	31b	31a	36b	36a	36c	39b	39a
29P	24d	27a	27c	27b	28a	30b	30a	31b	31a	36b	36a	36c	39b	39a
31P	24d	27a	27b	27c	28a	30a	30b	1a	1a	36a	36b	36c	39b	39a
37P	24c	27a	27b	27c	28a	30b	30a	31b	31a	36a	36b	36c	39b	39a
X.1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.2	−1	−2	1	1	−1	−1	−1	.	.	.	.	.	−1	−1
X.3	−1	2	−1	−1	−1	.	.	−1	−1	.	.	.	.	.
X.4	.	−1	−1	−1	.	2	2	.	.	−1	−1	−1	.	.
X.5	.	−1	−1	−1	1	−1	−1	.	.	1	1	1	.	.
X.6	1	.	.	.	.	−1	−1	.	.	−1	−1	−1	.	.
X.7	.	.	.	.	2	.	.	.	.	.	.	.	1	1
X.8	.	.	.	.	−1	.	.	.	.	.	.	.	.	.
X.9	.	1	1	1	.	.	.	.	.	−1	−1	−1	−1	−1
X.10	−1	.	.	.	1	.	.	.	.	1	1	1	.	.
X.11	1	.	.	.	−1	.	.	.	.	.	.	.	.	.

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