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Generalizing the generalized Petersen graphs

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Abstract

The generalized Petersen graphs (GPGs) which have been invented by Watkins, may serve for perhaps the simplest nontrivial examples of "galactic" graphs, i.e. those with a nice property of having a *semiregular automorphism*. Some of them are also vertex-transitive or even more highly symmetric, and some are Cayley graphs. In this paper, we study a further extension of the notion of GPGs with the emphasis on the symmetry properties of the newly defined graphs.

Keywords: Generalized Petersen graphs; Vertex-transitivity; Semiregular automorphisms

1. Introduction

The Petersen graph (see Fig. 1) is certainly one of the most famous objects that graph theorists have come across. This graph is a counterexample to many conjectures: for example, it is not 1-factorizable despite being cubic and without bridges (Tait's conjecture), and it is not hamiltonian. But being 3-transitive (that is, its automorphism group is transitive on directed paths of length 3), it is highly symmetric; however, it is not a Cayley graph! Many additional facts about the Petersen graph can be found in [4,8].

In [13], Watkins introduced the notion of *generalized Petersen graph* (GPG for short) as follows: given integers $n \geqslant 3$ and $k \in \mathbb{Z}_n - \{0\}$, the graph $P(n,k)$ is defined on the set $\{x_i, y_i \mid i \in \mathbb{Z}_n\}$ of $2n$ vertices, with the adjacencies given by $x_i x_{i+1}$, $x_i y_i$, $y_i y_{i+k}$ for all i. In this notation, the (classical) Petersen graph is $P(5,2)$. The GPGs have been studied by several authors; for instance, the complete classification of their (full) automorphism groups has been worked out in [6], while Nedela and Škoviera have determined all Cayley graphs among GPGs [12]. In the next section, we will describe a further generalization of $P(n,k)$ which preserves the property of having a *semiregular automorphism*, that is, an automorphism having all orbits of the same length, and the rest will be devoted to automorphism groups of these graphs. Let us mention a conjecture tying semiregular automorphisms to vertex-transitive graphs, and therefore to this paper as well. In [10], Marušič asked if it is true that every vertex-transitive graph has a semiregular automorphism. Several papers on this subject have recently appeared in the literature [2,5,7,11], but the problem is still open.

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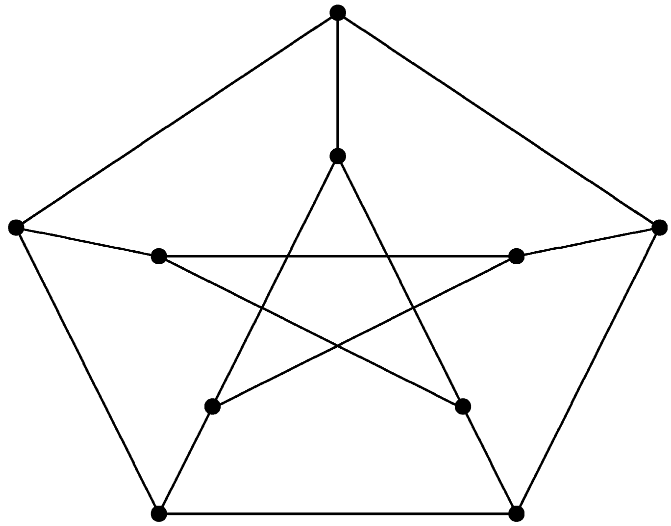

Fig. 1. The Petersen graph.

2. The supergeneralization

Definition 1. Let $m \geqslant 2$, $n \geqslant 3$ be integers and $k_0, k_1, \ldots, k_{m-1} \in \mathbb{Z}_n - \{0\}$. The vertex-set of the graph $P(m, n; k_0, \ldots, k_{m-1})$ is $\mathbb{Z}_m \times \mathbb{Z}_n$ and the edges are defined by $(i, j) \sim (i + 1, j)$, $(i, j) \sim (i, j + k_i)$, for all $i \in \mathbb{Z}_m$ and $j \in \mathbb{Z}_n$. The edges of type $(i, j) \sim (i + 1, j)$ will be called *horizontal*, while those of type $(i, j) \sim (i, j + k_i)$ *vertical*.

We will call such a graph *supergeneralized Petersen graph* (SGPG). As a subclass of SGPGs we find all GPGs, since $P(n, k)$ is actually isomorphic to $P(2, n; 1, k)$. $P(m, n; 1, \ldots, 1)$ is the Cartesian product $C_m \times C_n$ of two cycles; in particular, the skeleton of the 4-dimensional hypercube is $Q_4 = C_4 \times C_4 = P(4, 4; 1, 1, 1, 1)$. Another nontrivial example is given in Fig. 2.

Remark 2. Observe that the replacement of the parameter k_i by $n - k_i$ results in the same graph. Therefore, we will introduce an equivalence relation E on $\mathbb{Z}_n$, setting aEb if and only if $a = \pm b$, and consider the parameters k_i as E-classes representatives.

To avoid confusion we will pick $0 \leqslant k_i \leqslant n/2$ as a representative of $\{\pm k_i\}$. If n is odd, then $P = P(m, n; k_0, \ldots, k_{m-1})$ is obviously 4-regular, with the exception $m = 2$, when the graph is cubic. The same holds when n is even, if $k_i \neq n/2$ for every $i \in \mathbb{Z}_m$. On the contrary, if $k_i = n/2$ for some i, then P is not regular, unless $k_i = n/2$ for all $i \in \mathbb{Z}_n$. In this last case the graph P is not connected: it is formed by $n/2$ components each isomorphic to $C_m \times K_2$.

Definition 3. We say that for a fixed j, the subgraph $Z_j = \langle\{(i, j) : i \in \mathbb{Z}_m\}\rangle$ is an *horizontal level*, and for a fixed i, $W_i = \langle\{(i, j) : j \in \mathbb{Z}_n\}\rangle$ is a *vertical level*. We set $\mathscr{Z} = \{Z_j : j \in \mathbb{Z}_n\}$ and $\mathscr{W} = \{W_i : i \in \mathbb{Z}_m\}$.

Notice that every horizontal level is isomorphic to a cycle C_m. It is easy to see that there is a dihedral subgroup of $\mathrm{Aut}(P)$ acting transitively on horizontal levels.

Proposition 4. *Let $P = P(m, n; k_0, \ldots, k_{m-1})$ be a SGPG. Define $\alpha(i, j) = (i, j + 1)$ and $\beta(i, j) = (i, -j)$ for all i, j. Then $\alpha, \beta \in \mathrm{Aut}(P)$ and $B = \langle\alpha, \beta\rangle$ is a dihedral group of order $2n$ which acts transitively on horizontal levels and fixes every vertical level.*

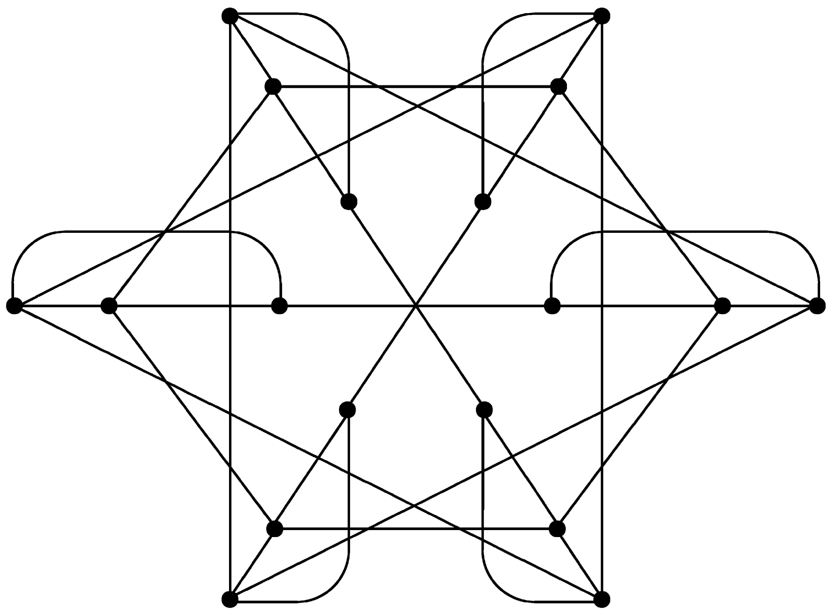

Fig. 2. $P(3, 6; 2, 1, 3)$.

Proof. Immediate. □

Now we would like to find out which conditions guarantee the connectedness of P in general. We recall that $a \equiv_n b$ will denote the more standard $a \equiv b \pmod n$.

Proposition 5. *Set $d = \gcd(n, k_0, k_1, \ldots, k_{m-1})$. Then $P = P(m, n; k_0, \ldots, k_{m-1})$ has d connected components each isomorphic to $P(m, n/d; k_0/d, \ldots, k_{m-1}/d)$. In particular, P is connected if and only if $d = 1$.*

Proof. Set $n' = n/d$ and consider the following table of cycles Z_j:

$$\begin{array}{cccc} Z_0 & Z_1 & \cdots & Z_{d-1} \\ Z_d & Z_{d+1} & \cdots & Z_{2d-1} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{(n'-1)d} & Z_{(n'-1)d+1} & \cdots & Z_{n'd-1}. \end{array} \tag{1}$$

We will show that every pair of cycles in the same column in (1) is connected by a path, while this is not true for pairs of cycles in different columns. There is a path connecting the cycle Z_0 with Z_j when

$$j \equiv_n \sum_{i=0}^{m-1} c_i k_i. \tag{2}$$

In fact, a path connecting the vertices $(i, 0) \in V(Z_0)$ and $(i', j) \in V(Z_j)$ necessarily contains vertical edges. Using a number-theoretical result, (2) holds if and only if j is divisible by d. Hence Z_0 is tied only to the cycles Z_{hd} $(0 \leqslant h < n')$ in the same column of (1). Since $\alpha(Z_j) = Z_{j+1}$, α permutes the columns of (1) cyclically and the assertion follows. □

The above defined automorphism α is a semiregular one. Recall that such an automorphism has all his orbits of the same length and we shall see how it is involved in a nice characterization of 4-regular SGPGs. First, we need a definition.

Definition 6. Let α be an automorphism of a graph Γ and Z an induced cycle of Γ. Then Z is said to be α-*transversal* if the following two conditions are met:

- $Z \cap \mathbb{O}$ is a single vertex for every orbit $\mathbb{O}$ of α,
- the neighbourhood of the only vertex of $Z \cap \mathbb{O}$ is contained in $Z \cup \mathbb{O}$.

Proposition 7. *Let Γ be a regular graph of degree* 4. *Then Γ is a SGPG if and only if it admits a nontrivial semiregular automorphism α and an α-transversal cycle Z.*

Proof. If $\Gamma = P(m, n; k_0, \ldots, k_{m-1})$ is 4-regular, then α and Z_0 as defined in the proof of Proposition 5 will do. Conversely, assume that such α and Z exist in a 4-regular graph Γ. Denote the orbits of α by $\mathbb{O}_0, \ldots, \mathbb{O}_{m-1}$. By definition of α, all orbits have the same length n (say) and there are n α-transversal cycles $Z, \alpha(Z), \ldots, \alpha^{n-1}(Z)$. This implies that every orbit $\mathbb{O}_i$ induces a 2-regular subgraph which is a disjoint union of cycles of equal length. Denote the vertex in $\mathbb{O}_i \cap \alpha^j(Z)$ by $x_j^{(i)}$. All adjacencies in the orbit $\mathbb{O}_i$ are completely determined by the neighbours $x_{k_i}^{(i)}$ and $x_{-k_i}^{(i)}$ of $x_0^{(i)}$, where $k_i \in \mathbb{Z}_n - \{0\}$ is characteristic for $\mathbb{O}_i$. Clearly $\Gamma = P(m, n; k_0, \ldots, k_{m-1})$ is an SGPG. $\square$

In view of this characterization, let us note that the SGPGs can be regarded as a subclass of (m, n)-*galactic graphs*, as defined by Marušič in [10]. As we shall see, some SGPGs, but certainly not all of them, are adorned with enough automorphisms to make them *uniformly* (m, n)-*galactic*. We now characterize the automorphisms preserving both $\mathscr{W}$ and $\mathscr{Z}$, the sets of horizontal and vertical levels, respectively.

Definition 8. As usual in permutation group theory, given a group A and a subset S of an A-set, we denote by $N_A(S) = \{a \in A \mid S^a = S\}$ and $C_A(S) = \{a \in A \mid s^a = s, \ \forall s \in S\}$ the setwise and pointwise stabilizer of S, respectively.

Theorem 9. *Let $A = \mathrm{Aut}(P)$, P a connected SGPG. Then, up to a power of α, any $\phi \in N_A(\mathscr{W}) \cap N_A(\mathscr{Z})$ has shape*:

1. $\phi(i, j) = (i + a, \lambda j)$, *where $m = ab$ in $\mathbb{Z}$ and $\lambda^{2b} \equiv_n 1$ or*
2. $\phi(i, j) = (-i + a, \lambda j)$, *where $a \in \mathbb{Z}_m$ and $\lambda^4 \equiv_n 1$.*

Proof. By assumption $\phi(Z_j) = Z_{\pi(j)}$ and $\phi(V_i) = V_{\sigma(i)}$, where $\pi \in \mathrm{Sym}(\mathbb{Z}_n)$ and $\sigma \in \mathrm{Sym}(\mathbb{Z}_m)$. Thus $\phi(i, j) = (\sigma(i), \pi(j))$. Denote by $N_{i,j}$ the neighbourhood of (i, j) in P. Looking at the ϕ-image of $N_{i,j} \cap Z_j$, we obtain that $\sigma(i+1) = \sigma(i) + \varepsilon_i$, where $\varepsilon_i = \pm 1$. Moving along Z_j we see that $\varepsilon_i = \varepsilon$ is independent from i, so $\sigma(i+h) = \sigma(i) + \varepsilon h$ for any h. Shifting to $N_{i,j} \cap V_i$, we get $\pi(j + k_i) = \pi(j) + \eta_i k_{\sigma(i)}$, where $\eta_i = \pm 1$. Again, it follows that for every h, $\pi(j + hk_i) = \pi(j) + \eta_i hk_{\sigma(i)}$. If $j = \sum_{i=0}^{m-1} v_i k_i$, $v_i \in \mathbb{Z}$, then

$$\pi(j) = \pi\left(\sum_{i=0}^{m-1} v_i k_i\right) = \pi(0) + \sum_{i=0}^{m-1} \eta_i v_i k_{\sigma(i)}.$$

Composing with $\alpha^{-\pi(0)}$, we may assume that $\pi(0) = 0$. By connectedness, there exist $u_0, \ldots, u_{m-1} \in \mathbb{Z}$ such that $\sum_{i=0}^{m-1} u_i k_i = 1$. Thus

$$\pi(j) = \pi\left(\sum_{i=0}^{m-1} j u_i k_i\right) = \sum_{i=0}^{m-1} \eta_i j u_i k_{\sigma(i)} = \pi\left(\sum_{i=0}^{m-1} u_i k_i\right) \cdot j = \pi(1) \cdot j \tag{3}$$

for any j. Setting $\lambda = \sum_{i=0}^{m-1} \eta_i u_i k_{\sigma(i)} = \pi(1)$, we get that π corresponds to the multiplication by λ. On the other hand

$$\pi(k_i) = \eta_i k_{\sigma(i)}.$$

Assume first that $\varepsilon = 1$. Then $\sigma(i) = i + \sigma(0)$. If $u\sigma(0) + vm = \gcd(\sigma(0), m)$, then $\langle \sigma \rangle = \langle \sigma^u \rangle$. So we may always reduce to the case where $a = \sigma(0)$ is a divisor of m, say $m = ab$. Hence $\Sigma = \langle \sigma \rangle$ has a orbits of length b, namely $i^{\Sigma} = \{i + ra \mid 0 \leqslant r < b\}$, $0 \leqslant i < a$. Thus

$$k_{i+ra} = \pm \lambda^r k_i.$$

In particular, $\lambda^b \pm 1$ annihilates k_i. Thus $(\lambda^{2b} - 1)k_i \equiv_n 0$. Since the k_i's are coprime in $\mathbb{Z}_n$, we get $\lambda^{2b} \equiv_n 1$.

If $\varepsilon = -1$, then σ is an involution. Arguing as before we get $(\lambda^2 \pm 1)k_i \equiv_n 0$ for any i, hence $\lambda^4 \equiv_n 1 \pmod n$. □

We draw some conclusions from the previous proof.

Definition 10. Define $\phi_{a,\lambda}$ by $\phi_{a,\lambda}(i, j) = (i + a, \lambda j)$ and $\psi_{a,\lambda}$ by $\psi_{a,\lambda}(i, j) = (-i + a, \lambda j)$ where $a \in \mathbb{Z}_m - \{0\}$, $\lambda \in \mathbb{Z}_n - \{0\}$.

Notice that the identity automorphism is $\phi_{0,1}$ and $\beta = \phi_{0,-1}$.

Corollary 11. *Let $ab = m$ and $\lambda^{2b} \equiv_n 1$. Then $\phi_{a,\lambda} \in \mathrm{Aut}(P(m, n; k_0, \ldots, k_{m-1}))$ if and only if $k_{i+ra} = \pm\lambda^r k_i$ where $0 \leqslant r < b$, $0 \leqslant i < a$. In particular, $\gcd(n, k_{i+ra}) = \gcd(n, k_i)$.*

Corollary 12. *Let $\lambda^4 \equiv_n 1$. Then $\psi_{a,\lambda} \in \mathrm{Aut}(P(m, n; k_0, \ldots, k_{m-1}))$ if and only if $k_{a-i} = \pm\lambda k_i$, $i \in \mathbb{Z}_m$. In particular, $\gcd(n, k_{a-i}) = \gcd(n, k_i)$.*

Proposition 13. *Let $A = \mathrm{Aut}(P(m, n; k_0, \ldots, k_{m-1}))$. Then*

1. *$\Phi = \{\phi_{a,\lambda} \in A\}$ is an abelian subgroup of A.*
2. *$\alpha^{\phi_{a,\lambda}} = \alpha^{\psi_{a,\lambda}} = \alpha^{\lambda^{-1}}$.*
3. *$\psi_{b,\mu} \in N_A(\Phi)$, in particular $\phi_{a,\lambda}^{\psi_{b,\mu}} = \phi_{-a,\lambda}$.*
4. *If $d = \gcd\{a : \phi_{a,\mu} \in A\}$, then $\phi_{d,\lambda} \in A$ for some λ.*
5. *If $\phi_{d,\lambda}$ and $\phi_{d,\mu}$ belong to A, then $\mu \equiv_n w\lambda$, where $w^2 \equiv_n 1$.*
6. *$N = N_A(\mathscr{W}) \cap N_A(\mathscr{Z})$ has a normal chain $1 \lhd \langle\alpha\rangle \unlhd \langle\alpha\rangle \rtimes \Phi \unlhd N$ with cyclic, abelian, and elementary abelian factors. In particular, N is always soluble.*

Proof. The first three claims follow from direct calculation. For the fourth, assume that $d = \sum_{j=1}^{t} u_i a_i$, where $\phi_{a_i,\lambda_i} \in A$. Then $\phi_{d,\lambda} = \prod \phi_{a_i,\lambda_i}^{u_i}$ where λ belongs to the (multiplicative) group generated by the λ_i's. For the fifth, it follows from Corollary 11 that $k_{i+dr} = \pm\lambda^r k_i = \pm\mu^r k_i$, so any k_i is either annihilated by $\lambda - \mu$ or by $\lambda + \mu$. Thus, $\lambda^2 = \mu^2$. Notice that if some k_i is coprime to n then $\mu = \pm\lambda$. The last claim follows from $\psi_{a,\lambda}^2 = \phi_{0,\lambda^2}$, so the last factor has exponent 2. □

Corollary 14. *$G = \langle\alpha, \beta, \phi_{a,\lambda}\rangle$ acts transitively on vertices if and only if $\gcd(a, m) = 1$.*

Proof. If $d = \gcd(a, m)$, then G has d orbits of length m/d on $\{W_0, \ldots, W_{m-1}\}$. If $d = 1$, the transitive action of $\langle\alpha, \beta\rangle$ on every W_i ensures that G is vertex-transitive. □

If $\gcd(a, m) = 1$, let t satisfy $at = 1$ in $\mathbb{Z}_m$. Then $\phi_{a,\lambda}^t = \phi_{1,\lambda^t}$. Thus all W_i's lie in the same $\phi_{a,\lambda}$-orbit and so $\bar{k}_0 = \cdots = \bar{k}_{m-1} = 1$. We may assume that $k_0 = 1$. Hence $k_i = \pm\lambda^i$ and $\lambda^m \equiv_n 1$ which implies that the graph

$$P(m, n; 1, \lambda, \lambda^2, \ldots, \lambda^{m-1}) \quad \text{with } \lambda^m \equiv \pm 1 \pmod n \tag{4}$$

is vertex-transitive. Alternatively, it can be regarded as a *metacirculant* $\mathrm{MC}(m, n, \lambda; S_0, S_1, \ldots, S_{m-1})$ where $S_0 = \{\pm 1\}$, $S_1 = \cdots = S_{m-1} = \{0\}$, using the notation of [1].

Are there any vertex-transitive SGPGs not obtained this way? More generally, the question is whether or not there exist SGPGs with primitive automorphism groups. As shown by the two cases $m = 2$ and 3 covered in this article, it is likely that such graphs are very rare.

3. Automorphisms of $P(2, n; k, l)$

Frucht et al. [6] have classified the groups $\mathrm{Aut}(P(n, k))$ for all values of n and k. Since according to our notation $P(n, k) = P(2, n; 1, k)$, it is natural to try to determine $\mathrm{Aut}(P(2, n; k, l))$ for $k, l > 1$. In what follows, we will write $\gcd(n, k) = \bar{k}$ and $\gcd(n, l) = \bar{l}$ for a fixed n; the dihedral group of $2n$ elements will be denoted by D_{2n}, and $\rtimes$ will stand for the usual semidirect product of groups.

We begin by pointing out a simple but useful fact.

Proposition 15. *If n, k (or n, l) are coprime, then $P(2, n; k, l)$ is isomorphic to $P(n, k^{-1}l)$ (or $P(n, l^{-1}k)$), where j^{-1} denotes the inverse of j in the ring $\mathbb{Z}_n$.*

Proof. Suppose that n, k are coprime. Then $(0, j) \mapsto x_{k^{-1}j}$ and $(1, j) \mapsto y_{k^{-1}j}$ describe the isomorphism between the two graphs. □

Hence we have to determine the groups $\mathrm{Aut}(P(2, n; k, l))$ only when n, k and n, l are not coprime. To assure connectedness, we assume that $\gcd(n, k, l) = 1$ (see Proposition 5). In short, our problem reduces to the determination of $A = \mathrm{Aut}(P)$ for the graphs $P = P(2, n; k, l)$ with

$$0 < k < l \leqslant n/2 \quad \& \quad \gcd(n, k) = \bar{k} > 1 \quad \& \quad \gcd(n, l) = \bar{l} > 1 \quad \& \quad \gcd(n, k, l) = 1. \tag{5}$$

Imitating the technique exploited in [6], we determine the 8-cycles containing at least one horizontal edge, a tool that allows us to verify the following

Proposition 16. *Every automorphism of P stabilizes the set of horizontal edges.*

Proof. We count the number of 8-cycles containing at least one horizontal edge. Obviously every cycle has to contain an even number of horizontal edges. If these were two, the cycle would have the form of a horizontal edge, followed by a vertical ones, followed by one horizontal edge again and finished by b vertical edges, where $\{a, b\} = \{1, 5\}, \{2, 4\}$ or $\{3\}$. This would imply that $ak \equiv bl \pmod{n}$, against $\gcd(n, k, l) = 1$. Since there are no consecutive horizontal edges, the 8-cycle possesses four horizontal edges and four vertical ones, alternatively. Thus, the order of its vertices is

$$y_j, x_j, x_{j\pm k}, y_{j\pm k}, y_{j\pm k\pm l}, x_{j\pm k\pm l}, x_{j\pm k\pm l\pm k}, y_{j\pm k\pm l\pm k}, y_{j\pm k\pm l\pm k\pm l} = y_j$$

for some j. But since both $2(l \pm k) = n$ and $2k = n$ contradict (5), there is essentially only one possible choice:

$$y_j, x_j, x_{j+k}, y_{j+k}, y_{j+k+l}, x_{j+k+l}, x_{j+l}, y_{j+l}.$$

For $j = 0$, denote this 8-cycle by R (see Fig. 3). Now $\alpha^{k+l}\beta$ stabilizes R and if $l < n/2$, then there are exactly n 8-cycles containing horizontal edges, namely $R, \alpha(R), \ldots, \alpha^{n-1}(R)$. Every horizontal edge belongs to exactly four of them and every vertical one to exactly two of them, and eventually to one more that contains only vertical edges. Hence there can be no automorphisms exchanging vertical and horizontal edges. The situation is somewhat different if $l = n/2$ because there are only $n/2$ 8-cycles with horizontal edges in this case. However, it is now the nonregularity of P that prevents any automorphism from exchanging vertical and horizontal edges. □

Thus A stabilizes both vertical and horizontal levels, and therefore we may apply Theorem 9. Notice that $\psi_{a,\lambda} = \phi_{a,\lambda}$ since $m = 2$. We recall that $\phi_{0,1} = 1_A$ and $\phi_{0,-1} = \beta$.

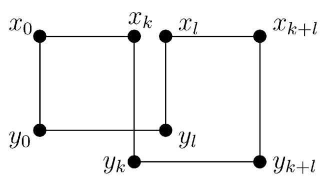

Fig. 3. The only 8-cycle.

Proposition 17. *There exists $\phi_{a,\lambda} \in A\backslash\{1_A, \beta\}$ if and only if $a=0$, $\lambda k \equiv_n \pm k$, $\lambda l \equiv_n \mp l$, $\lambda \not\equiv_n \pm 1$, $2l \neq n$, and $n=\bar{k}\bar{l}$ or $2\bar{k}\bar{l}$. Moreover, λ is unique up to sign.*

Proof. By conditions (5) $n = w\bar{k}\bar{l}$, for $w \in \mathbb{N}$. Proposition 4 requires that $a=0$. Since $\lambda \not\equiv_n \pm 1$, up to a sign change, we may assume that $\lambda k \equiv_n k$ and $\lambda l \equiv_n -l$ (notice that $2l=n$ would imply $\lambda = 1$). But these conditions are equivalent to $\lambda \equiv_{w\bar{l}} 1$ and $\lambda \equiv_{w\bar{k}} -1$. Then $2 \equiv_w 0$ and $w=1,2$. Assume now that $n = w\bar{k}\bar{l}$, with $w=1,2$ and $2l \neq n$. We slightly modify the usual proof of the Chinese remainder theorem. Namely define ρ from $\mathbb{Z}_n$ to $\mathbb{Z}_{w\bar{k}} \times \mathbb{Z}_{w\bar{l}}$ via $\rho(x) = (x_1, x_2)$, where $x_1 \equiv_{w\bar{k}} x$ and $x_2 \equiv_{w\bar{l}} x$.Then ρ is a ring monomorphism and $(x_1, x_2) \in \pi(\mathbb{Z}_n)$ if and only if $x_1 \equiv_w x_2$. So λ is uniquely determined as $\pi^{-1}(-1, 1)$. Notice that $-1 \equiv_{w\bar{k}} 1$ would force $w\bar{k} = 1, 2$ and $2l = n$, hence $\lambda \not\equiv_n \pm 1$. $\square$

Theorem 18. *Let $P=P(2,n;k,l)$ be a SGPG with $\gcd(n,k)=\bar{k}>1$, $\gcd(n,l)=\bar{l}>1$, $\gcd(n,k,l)=1$. Set $A=\mathrm{Aut}(P)$. Then $A=\langle\alpha,\beta\rangle \cong D_{2n}$, unless $n=w\bar{k}\bar{l}$, $w=1,2$, and $2l \neq n$ in which case $A=\langle\alpha,\beta,\phi_{0,\lambda}\rangle \cong \mathbb{Z}_n \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$, where λ is uniquely determined by $(\lambda-1)k \equiv_n (\lambda+1)l \equiv_n 0$.*

Proof. It follows from Proposition 17 that $A=\langle\alpha,\beta\rangle$, if the above condition does not hold. Otherwise $\exists\phi_{0,\lambda} \in A\backslash\langle\alpha,\beta\rangle$ and $A=\langle\alpha,\beta,\phi_{0,\lambda}\rangle$. Take i^*, j^* such that $ki^*+lj^* \equiv_n 1$; then $\lambda = ki^* - lj^*$. By Proposition 13, $\alpha^{\phi_{0,\lambda}} = \alpha^{\lambda}$ and $A \cong \mathbb{Z}_n \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$. $\square$

4. Automorphisms of $P(3, n; k_0, k_1, k_2)$

Let us consider an SGPG with three vertical levels, i.e. $P = P(3, n; k_0, k_1, k_2)$; denote the vertices $(0, j)$ by x_j, $(1, j)$ by y_j and $(2, j)$ by z_j, $j \in \mathbb{Z}_n$. The horizontal cycles Z_j are, in effect, triangles spanned by edges $x_j y_j$, $y_j z_j$, and $z_j x_j$. Also, there may exist triangles spanned by vertical edges. This happens for some i $(0 \leqslant i < 3)$ if and only if $n = 3k_i$. However, an arbitrary triangle in P must be either vertical or horizontal (cannot be of “mixed” type). This follows directly from the construction of P.

Proposition 19. *Let $P=P(3,n;k_0,k_1,k_2)$ and $A=\mathrm{Aut}(P)$. Then $A=N_A(\mathscr{Z})$ unless $P=P(3,3;1,1,1)=C_3 \times C_3$. In this case $N=N_A(\mathscr{Z})=N_A(\mathscr{W}) \cong D_6 \times D_6$ and $A=N \rtimes \mathbb{Z}_2$.*

Proof. It has been shown in the preceding paragraph that a triangle must be either horizontal or vertical. Suppose that $\phi(Z_j)$ is vertical. Then each of the vertices $\phi(x_j)$, $\phi(y_j)$ and $\phi(z_j)$ belongs to one vertical and one horizontal triangle. This implies that the same is true for the vertices x_j, y_j, and z_j, but then $n = 3k_i$ for all i and P is not connected by Theorem 5, unless $k_i = 1$ for all i and $n = 3$. The last claim can be checked using the package `nauty` as implemented in Magma (see [3] at http://magma.maths.usyd.au). $\square$

From now on we assume that $P \neq C_3 \times C_3$. Thus $N = N_A(\mathscr{W}) \cap N_A(\mathscr{Z}) = N_A(\mathscr{W})$. As an immediate consequence of Proposition 19, no automorphism in A maps a horizontal edge to a vertical one, nor vice versa. However, an automorphism ϕ may send a vertical cycle at one level to a cycle at another level. Yet this occurs only if the two cycles have the same size. Let us denote $\bar{k_i} = \gcd(n, k_i)$. In fact, $\bar{k_i}$ is the number of vertical cycles at level i, thence their size is equal to $n/\bar{k_i}$. This means, for example, that x-vertices can be mapped to y-vertices only if $\bar{k_0} = \bar{k_1}$. However, since P is connected, $\bar{k_0} = \bar{k_1} = \bar{k_2}$ is not possible unless all three $\bar{k_i}$ are equal to 1. Thus, we have to distinguish three possible situations:

(i) $\bar{k_0} = \bar{k_1} = \bar{k_2} = 1$,
(ii) $\bar{k_i}$ are all pairwise different,
(iii) two of $\bar{k_i}$ are equal and the third differs (say, $\bar{k_0} \neq \bar{k_1} = \bar{k_2}$).

Suppose that (i) holds. Then every level is a cycle. Since vertical cycles are sent to vertical cycles, we have $A = N_A(\mathscr{W})$ and may apply Theorem 9. Furthermore, we may assume that $k_0 = 1$. Notice that $\phi_{0,\lambda} \in A$ if and only if $\lambda \equiv_n \pm 1$ which is equivalent to $\phi_{0,\lambda} \in \{1_A, \beta\}$. Also, $\phi_{1,\lambda} \in A$ if and only if $k_1 = \lambda$, $k_2 = \pm\lambda^2$ with $\lambda^3 \equiv_n \pm 1$. Notice that $\lambda^3 \equiv_n -1 \not\equiv_n \lambda$ forces the order of $\phi_{1,\lambda}$ to be 6. Analogously $\psi_{b,\mu} \in A$ if and only if $k_{b-i} \equiv_n \pm \mu k_i$ (see Corollary 12). In particular $\mu \equiv_n \pm 1$. But $\psi_{b,-1} = \beta\psi_{b,1}$, so we may assume that $\mu = 1$. We now investigate when both type of automorphisms occur. Since $\mu = 1$, it follows that $k_2 = \pm k_1$ and $\lambda = \pm 1$. Since the k_i's are defined up to sign, we may assume that $\lambda = 1$ and $k_1 = k_2 = 1$. Since $\psi_{b,\mu}\psi_{c,\nu} = \phi_{b-c,\mu\nu}$, we see that either $\phi_{1,\lambda} \in A$, for some λ, or $\psi_{b,\mu} \in A$ for at most one b. Let us summarize these results in a theorem as follows.

Theorem 20. *Let $P = P(3, n; k_0, k_1, k_2)$ and $\gcd(n, k_i) = 1$ for all i. Then, up to graph isomorphism, we may assume that $k_0 = 1$. If we denote $A = \mathrm{Aut}(P(3, n; 1, k_1, k_2))$, then $A = N_A(\mathscr{W})$ except when $P = C_3 \times C_3$. Moreover, $A = \langle \alpha, \beta \rangle \cong D_{2n}$ except in the following cases*:

1. *$k_1 = k_2 = 1$, so $P = C_3 \times C_n$. If $n > 3$, then $C_A(\mathscr{W}) = \langle \alpha, \beta \rangle$, $C_A(\mathscr{Z}) = \langle \phi_{1,1}, \psi_{0,1} \rangle \cong D_6$, and $A = C_A(\mathscr{Z}) \times C_A(\mathscr{W}) \cong D_6 \times D_{2n}$. For $n = 3$ see Proposition* 19.
2. *$k_2 \not\equiv_n \pm k_1^2$ and there exists $b \in \mathbb{Z}_3$ such that $k_{-1-b} \equiv_n \pm k_{1-b}$. Then $A = \langle \alpha, \beta, \psi_{b,1} \rangle$ is isomorphic to $D_{2n} \times \mathbb{Z}_2$.*
3. *$1 \neq k_2 \equiv_n \pm k_1^2$ and $k_1^3 \equiv_n \pm 1$. Then $A = \langle \alpha, \beta, \phi_{1,k_1} \rangle$ is isomorphic to $\mathbb{Z}_n \rtimes \mathbb{Z}_6$.*

Let us make few remarks. The graphs in Cases 1 and 3 are *metacirculants* (see (4)), while in the remaining Case 2 they are not. It is also noteworthy that P of the previous theorem is vertex-transitive if and only if it is a metacirculant. However, only the exceptional graph $C_3 \times C_3$ is also edge-transitive (in fact, it is 1-transitive) and has a primitive automorphism group. Moreover, if P is a metacirculant, it is also a Cayley graph for the group $\langle \alpha, \phi_{1,\lambda} \rangle$ with the generating set $\{\alpha, \phi_{1,\lambda}\}$ where $k_1 = \pm\lambda$ and $\lambda^3 \equiv_n 1$. To see this, one has to put $x_j = \alpha^j$, $y_j = \phi_{1,\lambda}\alpha^{j\lambda^2}$ and $z_j = \phi_{1,\lambda}^2\alpha^{j\lambda}$, $j \in \mathbb{Z}_n$.

In situations (ii) and (iii) we determine the group $C_A(\mathscr{W})$, the pointwise stabilizer of vertical levels. Since horizontal levels coincide with horizontal triangles, this group is fully described by Theorem 9: $C_A(\mathscr{W})$ contains only automorphisms of the form $\phi_{0,\lambda}$, up to the powers of α. In case (ii), the vertical cycles in different vertical levels have different lengths, so $A = C_A(\mathscr{W})$. Since P is connected, it follows from Theorem 5 that every element of $\mathbb{Z}_n$ is expressible as $\sum_{r=0}^{2} k_r v_r$ $((\mathrm{mod}\, n))$, where $v_r \in \mathbb{Z}$. Then $\phi_{0,\lambda}(i, \sum_{r=0}^{2} k_r v_r) = (i, \lambda\sum_{r=0}^{2} k_r v_r) = (i, \sum_{r=0}^{2} \eta_r k_r v_r)$, where $\eta_r = \pm 1$. Up to multiplication by β, we may assume that $\eta_r = -1$ only for one r; say $r = 2$. Then $k_2 < n/2$, for otherwise $\phi_{0,\lambda}$ would be the identity. Now we can resort to Proposition 17 as follows. Let $\bar{k}_{01} = \gcd(n, k_0, k_1)$; then $\sum_{r=0}^{2} k_r v_r \equiv_n \bar{k}_{01}s + k_2 v_2$ for some s. Note that $\bar{k}_{01} \neq n/2$ necessarily, otherwise $\lambda = -1$ and $\phi_{0,\lambda} = \beta$. By Proposition 17, $\phi_{0,\lambda}$ is well defined if and only if

$$n = \bar{k}_{01}\bar{k}_2 \quad \text{or} \quad n = 2\bar{k}_{01}\bar{k}_2. \tag{6}$$

The example $n = 2pqr$, $k_0 = qr$, $k_1 = pr$ and $k_2 = pq$ where p, q, r are three distinct odd primes shows that (6) may hold for all three pairs. We show that if (6) holds for two pairs, it must hold for the third one as well. Assume that $n = a\bar{k}_{01}\bar{k}_2 = b\bar{k}_{02}\bar{k}_1$, where $\{a, b\} \subseteq \{1, 2\}$. Consider first the case $a = b = 1$: then $\bar{k}_1 = \bar{k}_{01}\bar{k}_{12}$ and $\bar{k}_2 = \bar{k}_{02}\bar{k}_{12}$. So $n = \prod_{i<j} \bar{k}_{ij}$. Since $\gcd(\bar{k}_0, \bar{k}_{12}) = 1$, it follows that $\bar{k}_0 = \bar{k}_{02}\bar{k}_{01}$ and $n = \bar{k}_{12}\bar{k}_0$. The same conclusion can be inferred for the remaining cases $\{a, b\} \subseteq \{1, 2\}$. So, if (6) holds for all three pairs, then there are three automorphisms ϕ_{0,λ_i}, where λ_i is an involution in $\mathbb{Z}_n$ satisfying $\lambda_j k_i = (-1)^{\delta_{ji}} k_i$; here δ_{ji} denotes the Kronecker delta. They generate a group isomorphic to $\mathbb{Z}_2^3$. So, we have proven

Theorem 21. *Assume that $|\{\gcd(n, k_i) \mid i \in \mathbb{Z}_3\}| > 1$ and $\gcd(n, k_0, k_1, k_2) = 1$. Then $B = C_A(\mathscr{W})$, where $A = \mathrm{Aut}(P(3, n; k_0, k_1, k_2))$, equals $\langle \alpha, \beta \rangle$ except when $k_0, k_1, k_2 \neq n/2$ and*

1. *n equals $\bar{k}_{ij}\bar{k}_h$ or $2\bar{k}_{ij}\bar{k}_h$ for only one choice of i, j, h such that $\{i, j, h\} = \{0, 1, 2\}$. Then $B = \langle \alpha, \beta, \phi_{0,\lambda_h} \rangle$ is isomorphic to $\mathbb{Z}_n \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$, where λ_h is the unique involution of $\mathbb{Z}_n$ such that $\lambda_h k_i = (-1)^{\delta_{hi}} k_i$;*
2. *n equals $\bar{k}_{ij}\bar{k}_h$ or $2\bar{k}_{ij}\bar{k}_h$ for at least two triples i, j, h such that $\{i, j, h\} = \{0, 1, 2\}$. Then $B = \langle \alpha \rangle \rtimes \langle \phi_{0,\lambda_h} \mid h \in \mathbb{Z}_3 \rangle$ is isomorphic to $\mathbb{Z}_n \rtimes \mathbb{Z}_2^3$ with action given by $\alpha^{\phi_{0,\lambda_h}} = \alpha^{\lambda_h}$, where $\lambda_h k_i = (-1)^{\delta_{hi}} k_i$.*

If the $\gcd(n, k_i)$'s are pairwise distinct, then $A = C_A(\mathscr{W})$.

What remains to delve into is the somewhat more difficult situation (iii). We first determine $N_A(\mathscr{W})$ by applying Theorem 9. Since the 0th level must be fixed, we only need to look for automorphisms of shape $\psi_{0,\mu}$, where $\mu^4 \equiv_n 1$.

By Corollary 12 we must have $k_2 \equiv_n \pm \mu k_1$, and $k_1 \equiv_n \pm \mu k_2$. By replacing μ with $-\mu$, that is multiplying by β, we may assume that

$$k_2 \equiv_n \mu k_1.$$

Notice that this condition uniquely determines μ modulo $n/\bar{k}_1$. Moreover, $(\mu - \eta_0)k_0 \equiv_n 0$ and $(\mu^2 - \eta_1)k_1 \equiv_n 0$, where $\eta_i = \pm 1$. To simplify notation we remark that $n = \bar{k}_0\bar{k}_1 w$, for $w \in \mathbb{N}$. Then $(\mu^2 - \eta_1)k_1 \equiv_n 0$ is equivalent to $\mu^2 \equiv_{w\bar{k}_0} \eta_1$, so that η_1 is uniquely determined by k_1 and k_2, and $(\mu - \eta_0)k_0 \equiv_n 0$ is equivalent to $\mu \equiv_{w\bar{k}_1} \eta_0$.

Assume first that $\eta_1 = 1$. Then η_0 can be chosen as 1 or -1 only if there exist involutions μ, v such that $\mu \equiv_{w\bar{k}_1} - v \equiv_{w\bar{k}_1} 1$ and $\mu \equiv_{w\bar{k}_1} v$. Thus $w|2$, forcing $n = \bar{k}_0\bar{k}_1$ or $n = 2\bar{k}_0\bar{k}_1$. Conversely let π denote the ring homomorphism from $\mathbb{Z}_n$ to $\mathbb{Z}_{w\bar{k}_0} \times \mathbb{Z}_{w\bar{k}_1}$ defined via $\pi(x) = (x \bmod w\bar{k}_0, x \bmod w\bar{k}_1)$. Since $\gcd(\bar{k}_0, \bar{k}_1) = 1$, π is a monomorphism and its image consists of all pairs (x_1, x_2) such that $x_1 \equiv_w x_2$. By Cauchy's Lemma there exists an involution $\mu_1 \in \mathbb{Z}^*_{w\bar{k}_0}$ (here, $\mathbb{Z}^*_{w\bar{k}_0}$ denotes the set of invertible elements of $\mathbb{Z}_{w\bar{k}_0}$) if and only if $2|\varphi(w\bar{k}_0)$ which is equivalent to $w\bar{k}_0 > 2$, or $2k_1 \neq n$, where φ denotes Euler totient function. Notice that μ_1 must be odd if $w = 2$. Thus we can choose μ and v as $\pi^{-1}(\mu_1, \pm 1)$.

Suppose now that $\eta_1 = -1$. Then $\mu^2 \equiv_{w\bar{k}_0} - 1$, $\mu \equiv_{w\bar{k}_1} \eta_0$, so $\mu^2 \equiv_{w\bar{k}_1} 1$. As before, $w|2$. Conversely let $n = w\bar{k}_0\bar{k}_1$, $w = 1, 2$. We investigate under which conditions there exists $\mu \in \mathbb{Z}^*_n$ satisfying the above conditions. Using again the map π we see that such μ exists if and only if there is a $\mu_1 \in \mathbb{Z}_{w\bar{k}_0}$ such that $\mu_1^2 \equiv_{w\bar{k}_0} - 1$. As it is well known, this holds if and only if every prime divisor of $\bar{k}_0$ is congruent to 1 modulo 4. Thus, $\mu = \pi^{-1}(\mu_1, \eta_0)$. Finally, $v = (-\mu)^3$ satisfies $v \equiv_{w\bar{k}_0} \mu$ and $v \equiv_{w\bar{k}_1} - 1$, hence all sign possibilities are exhausted by $\langle \beta, \psi_{0,\mu} \rangle$. We can now state the following result:

Theorem 22. *Suppose that* $\bar{k}_0 \neq \bar{k}_1 = \bar{k}_2$, $\gcd(\bar{k}_0, \bar{k}_1) = 1$. *Then* $n = w\bar{k}_0\bar{k}_1$, *for some* $w \in \mathbb{N}$. *Let* μ *be the unique integer modulo* $w\bar{k}_0$ *satisfying* $k_2 \equiv_n \mu k_1$. *Then* $N = N_A(\mathscr{W})$ *equals* $\langle \alpha, \beta \rangle$ *unless*

(a) $w \neq 1, 2$, $\mu \not\equiv_n \pm 1$, $\mu^2 \equiv_{w\bar{k}_0} 1$ *and* $\mu \equiv_{w\bar{k}_1} \eta_0$, $\eta_0 = \pm 1$. *Then* $N = \langle \alpha, \beta, \psi_{0,\mu} \rangle \cong \mathbb{Z}_n \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$.
(b) $w = 1, 2$, $\mu \not\equiv_n \pm 1$, $\mu^2 \equiv_{w\bar{k}_0} 1$, $\mu \equiv_{w\bar{k}_1} 1$. *Then there exists* v *such that* $v \equiv_{w\bar{k}_0} \mu$ *and* $v \equiv_{w\bar{k}_1} - 1$ *and* $N = \langle \alpha, \beta, \psi_{0,\mu}, \psi_{0,v} \rangle \cong \mathbb{Z}_n \rtimes \mathbb{Z}_2^3$. *Conversely, the existence of such* v *implies* $w = 1, 2$.
(c) $w = 1, 2$, $\mu^2 \equiv_{w\bar{k}_0} - 1$ *and* $\mu \equiv_{w\bar{k}_1} \eta_0$, $\eta_0 = \pm 1$. *Then* $N = \langle \alpha, \beta, \psi_{0,\mu} \rangle \cong \mathbb{Z}_n \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_4)$. *Conversely, such* μ *exists if and only if* $w = 1, 2$ *and* $\bar{k}_0$ *has only prime divisors congruent to* 1 *modulo* 4.

Proof. Most of the claims follow from the previous discussion. The exact structure of N is deduced from Proposition 13. □

Because the vertical cycles at levels 1 and 2 are now of the same size, suppose that there exists $\phi \in A$ exchanging them. (Note that the 0th level is always preserved.) Due to Proposition 19, to the presence of $\langle \alpha, \beta \rangle$ and to a possible renumeration of horizontal triangles we may argue that $\phi(x_{k_0 i}) = x_{k_0 i}$ for all i and $\phi(y_0) = z_0$, $\phi(z_0) = y_0$. Denote by W_0^x, W_0^y, W_0^z the vertical cycles through x_0, y_0, z_0, respectively. Thus, ϕ stabilizes W_0^x vertex-wise and interchanges W_0^y and W_0^z. It follows that

$$\phi(Z_{k_0 i}) = Z_{k_0 i} \quad \& \quad \phi(Z_{k_1 j}) = Z_{\pm k_2 j} \quad \& \quad \phi(Z_{k_2 l}) = Z_{\pm k_1 l} \tag{7}$$

for all i, j, l. Let $0 \leqslant p < \bar{k}_0$ and denote by W_p^x the vertical cycle through x_p. Since P is connected and (iii) holds, n, k_0 and k_1 are relatively prime, so $p \equiv_n k_0 i_p + k_1 j_p$ for some i_p, j_p. Hence $x_{k_1 j_p} = x_{p - k_0 i_p} \in W_p^x$ and $y_{k_1 j_p} \in W_0^y$ and it follows that $\phi(x_{k_1 j_p}) = x_{\pm k_2 j_p}$. Therefore, ϕ maps every vertex $x_{k_1 j_p + k_0 i} \in W_p^x$ to $x_{\pm k_2 j_p \pm k_0 i}$. But this is true for all W_p^x. Also, by replacing the pair k_0, k_1 with k_0, k_2 we realize that

$$\phi(x_{k_0 i + k_1 j}) = x_{k_0 i \pm k_2 j} \quad \& \quad \phi(x_{k_0 i + k_2 l}) = x_{k_0 i \pm k_1 l} \tag{8}$$

for arbitrary i, j, l. Recall that the sign at $k_0 i$ must be positive due to (7).

Suppose that there is a $y_{k_0 i + k_1 j}$ which is mapped by ϕ to another y-vertex. By (8), $\phi(y_{k_0 i + k_1 j}) = y_{k_0 i \pm k_2 j}$ and ϕ respects the adjacency structure only if $k_1 = k_2$. In this case there are $\bar{k}_1$ vertical cycles on either of the levels 1 and 2. Denote them by W_p^y and W_p^z, respectively, for each $0 \leqslant p < \bar{k}_1$. For every pair W_p^y, W_p^z there is a $\gamma_p \in A$ which swaps W_p^y and W_p^z perfectly along horizontal triangles, and fixes the rest of P. Clearly $\gamma_p^2 = \mathrm{id}$ and the γ_p's commute with each other. Also, $\beta\gamma_p\beta = \gamma_{\bar{k_1}-p}$ and $\alpha\gamma_p\alpha^{-1} = \gamma_{p+1}$ with the subscripts taken modulo $\bar{k_1}$. In particular, if our ϕ fixes every x-vertex, then it is a product of some γ_p's.

On the other hand, let ϕ be defined by (8) with the minus sign. If $\bar{k_1} = 1$ or $k_0 = n/2$, then (7) amounts to $\phi(Z_j) = Z_{-j}$ which means that ϕ is a product of β and some γ_p's. Therefore, suppose $\bar{k_1} > 1$; also, let $\bar{k_0} > 1$ since $\bar{k_0} = 1$ would lead us to essentially the same case described in the previous paragraph. Let Q be the subgraph of P induced by the edges $\{y_j z_j \mid j = 0, \ldots, n-1\}$ and $P' = P/Q$ (P contracted by Q). It follows that $P' \cong P(2, n; k_0, k_1)$ is connected and if B denotes the level-preserving subgroup of A, then $B \cong \mathrm{Aut}(P')$ due to Proposition 16. Moreover, ϕ gives rise to a δ' acting on P' and by Theorem 18, $\delta' \in \mathrm{Aut}(P')$ if and only if $n = \bar{k_0}\bar{k_1}$ or $n = 2\bar{k_0}\bar{k_1}$ with $k_0, k_1 \neq n/2$. If this is the case, then there is a $\delta \in B$ corresponding to $\delta' \in \mathrm{Aut}(P')$. Clearly $\delta\phi$ fixes every x-vertex and it follows that ϕ itself is a product of δ and some of the γ_p's. For a given $p \equiv k_0 i_p + k_1 j_p \pmod{\bar{k_1}}$ denote $p^* \equiv k_0 i_p - k_1 j_p \pmod{\bar{k_1}}$. It is not difficult to see that $\delta\gamma_p\delta = \gamma_{p^*}$, thus, we may state the following

Theorem 23. *Suppose that* $\bar{k}_0 \neq \bar{k}_1 = \bar{k}_2$, $\gcd(\bar{k}_0, \bar{k}_1) = 1$. *Then* $n = w\bar{k}_0\bar{k}_1$, *for some* $w \in \mathbb{N}$. *Let* λ *be the unique integer modulo* $w\bar{k}_0$ *satisfying* $k_2 \equiv_n \lambda k_1$. *Set* $A = \mathrm{Aut}(P(3, n; k_0, k_1, k_2))$.

- *If* $k_2 \not\equiv_n \pm k_1$, *then* $A = N_A(\mathscr{W})$. (*See Theorem* 22.)
- *If* $k_2 \equiv_n \pm k_1$, $w = 1, 2$, *and* $k_0, k_1 \neq n/2$, *then* $A = \langle \gamma_i, \alpha, \beta, \phi_{0,\lambda} \mid i \in \mathbb{Z}_{\bar{k}_1} \rangle \cong \mathbb{Z}_2^{\bar{k}_1} \rtimes (\mathbb{Z}_n \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2))$, *where* $\lambda \equiv_{w\bar{k}_1} 1$ *and* $\lambda \equiv_{w\bar{k}_1} \pm 1$.
- *If* $k_2 \equiv_n \pm k_1$, $w \neq 1, 2$, *then* $A = \langle \gamma_i, \alpha, \beta \mid i \in \mathbb{Z}_{\bar{k}_1} \rangle \cong \mathbb{Z}_2^{\bar{k}_1} \rtimes D_{2n}$.

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Further reading

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