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Global Cayley maps and conjugacy class sizes of maximal unipotent subgroups of finite simple groups of Lie type

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ABSTRACT

Let G_F be the group of points fixed by the Frobenius endomorphism F of a simple algebraic group G defined in characteristic p . In this paper, we show that if p is a good prime then the size of every U_F -conjugacy class, U_F a p -Sylow subgroup of G_F , is a q -power, where q is the level of F . We also determine the existence of power series inducing isomorphisms between U_F and its Lie algebra.

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1. Introduction

Let G be a simple affine algebraic group defined over the algebraic closure $\overline{\mathbb{F}}_p$ of a finite field \mathbb{F}_p . We may assume that G is a closed subgroup of $GL_d(\overline{\mathbb{F}}_p)$, for some d . Given a power q_0 of p , define $\sigma_{q_0}(x_{ij}) = (x_{ij}^{q_0})$. Then σ_{q_0} is an endomorphism of $GL_n(\overline{\mathbb{F}}_p)$, called a **standard Frobenius endomorphism**. An endomorphism F of G is called a **Frobenius endomorphism** if there exists an integer m such that $F^m(x) = \sigma_{q_0}(x)$, for all $x \in G$. We define $q := q_0^{1/m}$ as the **level** of F (see [9, Definition 2.1.9]). We denote by G_F , the subgroup of G fixed by F . G_F is called a finite simple group of Lie type.

The group G contains an F -stable Borel subgroup B which itself contains an F -stable maximal torus $T \leq B$. Let $U := R_u(B)$ denote the unipotent radical of B . Then U is also F -stable, and $U_F := G_F \cap U$ is a p -Sylow subgroup of G_F .

We say that a p -group P is a q -**power size** group if any conjugacy class of P has cardinality a power of q .

As is well-known, U_F is a q -power size group when p is large enough. Large usually means that the logarithm map is defined on U (cf. [17]). In [27], it was proved that any good prime is “large enough” when G is a classical group or of type G_2 . This led to the conjecture that U_F is a q -power size group when p is good without further restrictions on G . In this paper, we prove this conjecture.

Theorem A. *Let $U_F \leq G_F$ be a p -Sylow subgroup of the finite simple group of Lie type G_F , where G is defined over $\overline{\mathbb{F}}_p$. We assume further that p is a good prime for G . Then U_F is a q -power size group, where q equals the level of F .*

If p is a bad prime for G , then Theorem A will be false in general. To be more precise, for any pair (\mathcal{D}, p) , where \mathcal{D} is a Dynkin diagram and p a bad prime for \mathcal{D} , there is a pair (G, F) with G of type \mathcal{D} and U_F not a q -power size group. This is shown using the explicit determination of the class sizes obtained by several authors (see [8,26,25,27,28,33–35,42]). A more uniform argument uses the concept of F -conjugacy classes and considers **regular unipotent elements** (cf. [14, Section 8.5], [22,19,20]).

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Theorem A can be deduced easily from certain rationality properties concerning U_F (cf. Section 3) and the following consequence of Theorem 2.2.

Theorem 1.1. *Let G be a simple affine algebraic group defined over $\overline{\mathbb{F}}_p$, and let U be a maximal unipotent subgroup of G . Assume further that p is a good prime for G . Then $C_U(u)$ is connected for all $u \in U$.*

It should be noted that this result is false, if p is a bad prime for G (cf. [22], [40, Chap. III, (1.14d)]).

These results rely heavily on generalized Cayley maps. These maps (defined in the next section) were shown to exist by Springer. In the last part of the paper, we investigate under which conditions these maps may be realized by power series. In this case, we call them **global Cayley maps**. We recall that they exist when G is of classical type and p is good. For groups of exceptional types, we will show that essentially the only global Cayley map is a truncated exponential.

Using the Baker–Campbell–Hausdorff Formula, we refine some results from [27] extending the range of primes p for which the exponential map defined in characteristic 0 can be specialized to matrices with entries in a field of characteristic p . Moreover, we show that for groups of type G_2 and 3D_4 there exists a global Cayley map when $p \geq 5$.

Similar investigations were led by various authors (see [41,24,29,30,32]). Their purpose was to construct in good characteristic suitable algebraic groups containing given unipotent elements. The main technique relies on the so-called saturation, namely the possibility of defining a one-parameter group exploiting exponential maps that contains an assigned unipotent element.

We would like to point out that these investigations have revealed a mistake in [22] about the structure of $C_U(u)/C_U(u)^\circ$, when u a regular unipotent element in groups of type F_4 in characteristic 2 (see also [23]).

For most of the computations we relied on the symbolic computer system MAGMA (see [1]).

2. Centralizers in maximal unipotent subgroups

Let G be a simple affine algebraic group defined over an algebraic closure K of \mathbb{F}_p , and let \mathfrak{g} denote its K -Lie algebra. Let $B \leq G$ be a Borel subgroup of G , and let $U := R_u(B)$ denote its unipotent radical; in particular U is connected. It is well-known that U is a maximal unipotent subgroup of G , and that every unipotent subgroup of G is contained in a G -conjugate of U (cf. [11, Thm. 30.4]). By $\mathfrak{u} \leq \mathfrak{g}$ we denote the K -Lie algebra of U . In this section we study the centralizers of elements in U .

2.1. Generalized Cayley maps

A generalized Cayley map

$$f: U \longrightarrow \mathfrak{u} \tag{2.1}$$

is an isomorphism of algebraic varieties commuting with the (left) B -action. In [39, Proposition 3.5], T.A. Springer proved the following important result.

Theorem 2.1. *Suppose G is simply-connected and that p is a good prime for G . Then there exists a generalized Cayley map $f: U \rightarrow \mathfrak{u}$.*

Let $f: U \rightarrow \mathfrak{u}$ be a generalized Cayley map. Then $f(1) = 0$ (cf. [40, Chap. III, 3.13]). For $x \in U, X := f(x)$ and $t \in K$ one defines

$$x_t := f^{-1}(t.X). \tag{2.2}$$

In particular, $\{x_t \mid t \in K\}$ is a closed connected set containing x and 1. As the action of U commutes with f , one has

$$C_U(x) = C_U(x_t) \tag{2.3}$$

for all $t \in K^*$.

2.2. Centralizers in maximal unipotent subgroups

From Theorem 2.1 one concludes the following theorem:

Theorem 2.2. *Let p be a good prime for G , and let U be a maximal unipotent subgroup of G . Then $C_U(u)$ is connected for all $u \in U$.*

Proof. It suffices to prove the theorem for G simply-connected (cf. [40, Proof of Proposition III.3.15]). By Theorem 2.1, there exists a generalized Cayley map $f: U \rightarrow \mathfrak{u}$. For $t \in K^*$ and $x \in C_U(u)$ one has $u \in C_U(x) = C_U(x_t)$ (cf. (2.3)). In particular, $\{x_t \mid t \in K^*\} \subseteq C_U(u)$, and this implies that $x \in C_U(u)^\circ$. \square

Remark 2.3. Let p be a bad prime for G and let $u \in G$ be a regular unipotent element. It is well-known that $C_G(u)$ is a unipotent subgroup and thus $C_G(u)$ is contained in a maximal unipotent subgroup U of G . Since p is a bad prime for G , $u \notin C_G(u)^\circ = C_U(u)^\circ$. Thus generalized Cayley maps cannot exist in this case (cf. [40, Chap. III, (1.14d)]).

3. Rationality properties

In this section we assume that G_F is a finite simple group of Lie type not of type 2B_2 , 2G_2 , 2F_4 . By q we denote the level of F .

Let $B \leq G$ be an F -stable Borel subgroup of G , and let $T \leq B$ be an F -stable maximal torus contained in B . Let $U := R_u(B)$. In particular, U is also F -stable. By Φ we denote the root system of G with respect to (B, T) .

3.1. Smooth connected unipotent affine algebraic group schemes

Let \mathfrak{G} be an algebraic affine group scheme defined over a field K , i.e., $K[\mathfrak{G}]$ is a commutative K -Hopf algebra with antipode, $K[\mathfrak{G}]$ is a finitely generated commutative K -algebra, and \mathfrak{G} is represented by

$$\mathfrak{G}(S) := \text{Hom}_{K\text{-alg}}(K[\mathfrak{G}], S), \tag{3.1}$$

where S is a commutative K -algebra.

Let \bar{K} denote the algebraic closure of K , and let $\mathfrak{G}_{\bar{K}}$ denote the algebraic affine group scheme defined over \bar{K} which is represented by $\bar{K} \otimes_K K[\mathfrak{G}] = \bar{K}[\mathfrak{G}]$.

One calls \mathfrak{G} *reduced*, if $K[\mathfrak{G}]$ has no non-trivial nilpotent elements, and *smooth*, if $\mathfrak{G}_{\bar{K}}$ is reduced (cf. [43, Section 11.6]). In particular, if K is perfect, then every reduced algebraic affine group scheme is also smooth (cf. [43, Example 6.2]).

Let \mathfrak{G} be a smooth algebraic affine group scheme. Then \mathfrak{G} is *connected*, if and only if $K[\mathfrak{G}]$ is an integral domain (cf. [43, Section 6.6]). Hence, provided \mathfrak{G} is smooth, \mathfrak{G} is connected, if and only if $\mathfrak{G}(\bar{K})$ is connected as topological space in the Zariski topology (cf. [43, Section 6.5]).

A smooth algebraic affine group scheme \mathfrak{G} is *unipotent*, if every element in $\mathfrak{G}(\bar{K})$ is unipotent. For a smooth connected unipotent algebraic affine group scheme \mathfrak{G} defined over a perfect field K , the structure of $\mathfrak{G}(K)$ as K -algebra is well-known. More precisely, one has the following:

Theorem 3.1. *Let \mathfrak{G} be a smooth connected unipotent algebraic affine group scheme defined over the perfect field K . Then $K[\mathfrak{G}]$ is isomorphic as a K -algebra to $K[X_1, \dots, X_n]$, where $n = \dim(\mathfrak{G}(\bar{K}))$.*

Proof. Since \mathfrak{G} is a unipotent algebraic affine group scheme, \mathfrak{G} is a closed subgroup scheme of some unitriangular matrix group scheme (cf. [5, Prop. IV.2.2.5]). Since \mathfrak{G} is smooth and connected and as K is perfect, this implies that \mathfrak{G} is K -solvable (cf. [5, Proposition IV.4.3.4(b)]). The assertion thus follows from [5, Corollary IV.4.3.8]. \square

3.2. Rationality properties of U

We first assume that F acts trivially on the root system Φ of G . Then G_F is not of twisted type and thus a Chevalley group.

Let \leq be a total order of the set of positive roots Φ^+ such that $\alpha \leq \beta$ implies $\text{ht}(\alpha) \leq \text{ht}(\beta)$, where $\text{ht}: \Phi^+ \rightarrow \mathbb{N}_0$ denotes the height function with respect to the basis $\Delta \subseteq \Phi^+$. By the Lang-Steinberg Theorem and the fact that Borel subgroups in G coincide with their normalizer, F -stable maximal unipotent subgroups in G are G_F -conjugate. Thus we may assume that

$$\begin{aligned} U_F &= \left\{ \prod_{\alpha \in \Phi^+} x_\alpha(t_\alpha) \mid t_\alpha \in \mathbb{F}_q \right\}, \\ U &= \left\{ \prod_{\alpha \in \Phi^+} x_\alpha(t_\alpha) \mid t_\alpha \in \bar{\mathbb{F}}_q \right\}, \end{aligned} \tag{3.2}$$

where $x_\alpha(t_\alpha)$ is a root element in the root group U_α (cf. [3]), and the product is taken in increasing order (with respect to \leq).

Let $\mathbb{F}_q[\mathfrak{U}] := \mathbb{F}_q[X_\alpha \mid \alpha \in \Phi^+]$. Assigning X_α the value t_α yields an identification

$$\begin{aligned} U_F &= \text{Hom}_{\mathbb{F}_q\text{-alg}}(\mathbb{F}_q[\mathfrak{U}], \mathbb{F}_q), \\ U &= \text{Hom}_{\mathbb{F}_q\text{-alg}}(\bar{\mathbb{F}}_q[\mathfrak{U}], \bar{\mathbb{F}}_q). \end{aligned} \tag{3.3}$$

Moreover, the Chevalley Commutator Formula (cf. [3, Theorem 5.2.2]) and the commutator-collection-process show that there exist polynomials $f_\alpha \in \mathbb{F}_q[X_\gamma, Y_{\gamma'} \mid \alpha \succ \gamma, \gamma' \in \Phi^+]$, such that

$$\prod_{\alpha \in \Phi^+} x_\alpha(s_\alpha) \cdot \prod_{\alpha \in \Phi^+} x_\alpha(t_\alpha) = \prod_{\alpha \in \Phi^+} x_\alpha(s_\alpha + t_\alpha + f_\alpha(s_\gamma, t_{\gamma'})). \tag{3.4}$$

A similar argument applies for the inversion $\text{---}^{-1}: U \rightarrow U$. Thus with the comultiplication $\Theta: \mathbb{F}_q[\mathfrak{U}] \rightarrow \mathbb{F}_q[\mathfrak{U}] \otimes \mathbb{F}_q[\mathfrak{U}]$ corresponding to (3.4) and the antipode $S: \mathbb{F}_q[\mathfrak{U}] \rightarrow \mathbb{F}_q[\mathfrak{U}]$ corresponding to the inversion map, $\mathbb{F}_q[\mathfrak{U}]$ is a cocommutative \mathbb{F}_q -Hopf algebra with antipode, defining a smooth connected unipotent affine algebraic group scheme \mathfrak{U} over \mathbb{F}_q satisfying $\mathfrak{U}(\mathbb{F}_q) = U_F$, $\mathfrak{U}(\bar{\mathbb{F}}_q) = U$.

Assume now that G_F is of type ${}^2A_\ell, {}^2D_\ell, {}^3D_4$ or 2E_6 , and let m denote the order of the automorphism $\gamma \in \text{Aut}(\Phi)$ induced by F . Let w be **normal** for $\mathbb{F}_{q^m}/\mathbb{F}_q$, i.e. $\{w, \dots, w^{q^{m-1}}\}$ is a normal basis of $\mathbb{F}_{q^m}/\mathbb{F}_q$. Then the \mathbb{F}_q -algebra \mathbb{F}_{q^m} can be represented as \mathbb{F}_q -rational points of an \mathbb{F}_q -ring scheme $\mathfrak{E}, \mathbb{F}_q[\mathfrak{E}] = \mathbb{F}_q[X_1, \dots, X_m]$, i.e.,

$$\mathfrak{E}(R) = R \otimes_{\mathbb{F}_q} \mathbb{F}_{q^m} = R \oplus R.w \oplus R.w^q \oplus \dots \oplus R.w^{q^{m-1}}. \tag{3.5}$$

Moreover, every automorphism $\tau \in \text{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)$ is rational, and thus can be represented by an \mathbb{F}_q -algebra isomorphism $\tau_*: \mathbb{F}_q[\mathfrak{E}] \rightarrow \mathbb{F}_q[\mathfrak{E}]$.

Note that for $m = 2, \varepsilon \in \{\pm 1\}$ and $\{1, \tau\} = \text{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$ the subfunctor $\mathfrak{E}' \leq \mathfrak{E}^2$,

$$\mathfrak{E}'(R) := \{ (s, t) \in \mathfrak{E}(R) \times \mathfrak{E}(R) \mid s + s^\tau = \varepsilon t t^\tau \} \tag{3.6}$$

is closed and can be represented by the commutative \mathbb{F}_q -algebra $\mathbb{F}_q[X_1, X_2, X_3]$.

Let $\langle \gamma \rangle \cdot \Phi^+$ denote the set of $\langle \gamma \rangle$ -orbits on Φ^+ . From our hypothesis on (G, F) one concludes that every $\langle \gamma \rangle$ -orbit is of type $A_1, A_1 \times A_1, A_2$ or $A_1 \times A_1 \times A_1$. In particular, every root group $X_{\bar{\alpha}}, \bar{\alpha} \in \langle \gamma \rangle \cdot \Phi^+$, in U_F can be represented by a smooth connected unipotent affine algebraic group scheme $\mathfrak{X}_{\bar{\alpha}}$ defined over \mathbb{F}_q satisfying $\mathfrak{X}_{\bar{\alpha}}(\mathbb{F}_q) = X_{\bar{\alpha}}$ (cf. [3, Proposition 13.6.4,(i)–(iv)]). Thus using [3, Proposition 13.6.1] and the Chevalley Commutator Formula one can define a smooth connected unipotent affine algebraic group scheme \mathfrak{U} defined over \mathbb{F}_q satisfying $\mathfrak{U}(\mathbb{F}_q) = U_F$ and $\mathfrak{U}(\overline{\mathbb{F}}_q) = U$ as in the previous case.

3.3. Centralizers of F -fixed points

Let $\mathbb{F}_q[\mathfrak{U}]$ denote the smooth connected unipotent algebraic affine group scheme constructed in Section 3.2 satisfying (3.3). Let $u \in U_F$. Then u corresponds to a homomorphism of \mathbb{F}_q -algebras $r_u: \mathbb{F}_q[\mathfrak{U}] \rightarrow \mathbb{F}_q$. Let $r_c: \mathbb{F}_q[\mathfrak{U}] \rightarrow \mathbb{F}_q[\mathfrak{U}] \otimes \mathbb{F}_q[\mathfrak{U}]$ denote the morphism of \mathbb{F}_q -algebras representing the commutator map $c: U \times U \rightarrow U, c(x, y) = xyx^{-1}y^{-1}$.

Let $r_{c(u)} := (r_u \otimes id) \circ r_c: \mathbb{F}_q[\mathfrak{U}] \rightarrow \mathbb{F}_q[\mathfrak{U}]$ denote the morphism of \mathbb{F}_q -algebras corresponding to the mapping $c(u, _): U \rightarrow U$, and let I_u denote the radical of the ideal generated by $im(r_{c(u)})$. Then I_u is a \mathbb{F}_q -Hopf ideal, and the \mathbb{F}_q -Hopf algebra $\mathbb{F}_q[\mathfrak{U}]/I_u$ represents the reduced unipotent affine algebraic group scheme $C_{U(_)}(u)_{red}$ defined over \mathbb{F}_q (cf. [5, Corollary II.1.3.7], [11, Section 34.2]).

Theorem 3.2. *Let G_F be a finite simple group of Lie type, where G is defined over $\overline{\mathbb{F}}_p$, and assume that p is a good prime for G . Let $U \leq G$ be an F -stable maximal unipotent subgroup of G . If $u \in U_F$, then*

$$|C_{U_F}(u)| = q^{\dim(C_U(u))}, \tag{3.7}$$

where q equals the level of F .

Proof. We may assume that G is of adjoint type, and that U_F is equal to the \mathbb{F}_q -rational points of the group scheme $\mathbb{F}_q[\mathfrak{U}]$, i.e., $U_F = \mathfrak{U}(\mathbb{F}_q)$. By construction, $C_{U_F}(u)$ is equal to the \mathbb{F}_q -rational points of the reduced unipotent affine algebraic group scheme $\mathfrak{C} := C_{\mathfrak{U}(_)}(u)_{red}$ defined above. Moreover, $C_U(u) = \mathfrak{C}(\overline{\mathbb{F}}_q)$. This yields that the Krull dimension $\dim(\overline{\mathbb{F}}_q[\mathfrak{C}_{\overline{\mathbb{F}}_q}])$ equals $\dim(C_U(u))$.

Since \mathfrak{C} is reduced and as \mathbb{F}_q is perfect, \mathfrak{C} is smooth (cf. Section 3.1). Moreover, as $C_U(u)$ is connected (cf. Theorem 2.2), \mathfrak{C} is connected (cf. Section 3.1). Thus \mathfrak{C} is a smooth connected unipotent algebraic affine group scheme defined over \mathbb{F}_q , and Theorem 3.1 applies. This yields the claim. \square

4. Centralizers of regular unipotent elements in bad characteristic

Given a connected reductive group G we recall that regular unipotent elements exist and form a unique conjugacy class in G . Let F be a Frobenius endomorphism of G . Then G_F always possesses regular unipotent elements. If $Z = Z(G)$ is connected and p is a good prime, then these elements are conjugate in G_F to $u = \prod_{\alpha \in \Delta} x_\alpha(1)$, where Δ is a fundamental system for the root system Φ of G (see [4, Section 5.1]).

In [38, Lemma 4.3] Springer has proved that $C_G(u) = Z(G)C_U(u)$ for a regular unipotent element u . This simplifies theoretically the determination of $C_G(u)$, since we may just use the Chevalley Commutator Formula in U and reduce the problem to a system of equations in N variables, where $N = |\Phi^+|$ is the number of positive roots associated to G . In the same year Springer proved that $C_G(u)^\circ$ is abelian (see [37]). Lou, using inductive arguments and computer assistance, claimed to have proved that $C_G(u) = \langle u \rangle C_G(u)^\circ$ (see [22,19] for a converse). In particular, $C_G(u)$ is abelian. Since there is no track of these arguments, nor of any computer code, we have implemented MAGMA code in the exceptional cases and checked that, for bad primes $p, C_U(u) = \langle u \rangle C_U(u)^\circ$ and that $C_U(u)/C_U(u)^\circ$ is a cyclic group of order 4 when G is of type F_4 or $E_\ell, \ell = 7, 8$ and $p = 2$, and of order p otherwise. We point out that this agrees with Lou's results apart from type F_4 . In fact she claimed that $|C_U(u)/C_U(u)^\circ| = 2$ in this case. This result has been also recently obtained by Liebeck and Seitz (see [23]).

Assume A is a group admitting the action of F . We may define the **F -conjugacy class** of $a \in A$ as $\{baF(b)^{-1} : b \in A\}$. It turns out that F -conjugacy classes are related to the splitting of $g^G, g \in G_F$, into G_F -conjugacy classes, namely there is a bijection between the G_F -classes into which g^G splits and the F -conjugacy classes of $C/C^\circ, C = C_G(g)$ (see [14, Section 8.5]).

Fix a regular unipotent element u of G lying in G_F . Since $C = C_G(u)$ equals $\langle u \rangle C^\circ$, the F -conjugacy classes of C/C° , $H^1(F, C/C^\circ)$, have shape $\{u^i C^\circ\}$. Hence

$$|H^1(F, C/C^\circ)| = |C/C^\circ| = |u C^\circ|.$$

Observe that $v = {}^x u = x u x^{-1}$ lies in G_F iff $L(x) = x^{-1} F(x) \in C_G(u)$. We claim that ${}^x C_{G_F}(u) = C_{G_F}(v)$. We only need to check that ${}^x t \in G_F$ whenever $t \in C_{G_F}(u)$. Now $F({}^x t) = {}^{F(x)} t = {}^{xz} t = {}^x t$, where $z \in C_G(u)$, the last equality following from $C_G(u)$ being abelian. Notice that by Lang-Steinberg Theorem any element of $C_G(u)$ has shape $L(x)$ for some $x \in G$, so all G_F -conjugates of u are taken into account. It follows that u^G splits in G_F -classes of the same cardinality (see [7, Remark 14.22]).

Therefore we have proved the following.

Theorem 4.1. *Let u be a regular unipotent element of a simple affine algebraic group G stable under the Frobenius endomorphism F of level q , q a power of a bad prime p . Then $|u^{G_F}| = q^{N-\ell}/t$, where N is the number of positive roots, ℓ the semisimple rank of G , and $t = |H^1(F, C_G(u)/C_G(u)^\circ)|$ is either p or 4 , the latter occurring when $p = 2$.*

In particular, the class size of u in G_F is not an integral power of q , unless $q = p$ or a divisor of 4 .

These results are confirmed by the explicit analysis of conjugacy classes in finite groups of Lie type. Following [9, Section 2.2] we distinguish three cases for G_F :

- (a) a Chevalley group $A_\ell(q)$, $B_\ell(q)$ ($\ell \geq 2$), $C_\ell(q)$ ($\ell \geq 3$), $D_\ell(q)$ ($\ell \geq 4$), $G_2(q)$, $F_4(q)$, $E_\ell(q)$ ($\ell = 6, 7, 8$);
- (b) a Steinberg group ${}^2A_\ell(q)$ ($\ell > 1$), ${}^2D_\ell(q)$ ($\ell \geq 3$), ${}^3D_4(q)$, ${}^2E_6(q)$;
- (c) a Ree–Suzuki group ${}^2B_2(2^{a+\frac{1}{2}})$, ${}^2G_2(3^{a+\frac{1}{2}})$, ${}^2F_4(2^{a+\frac{1}{2}})$, $a \geq 0$.

4.1. Chevalley groups

A_ℓ has no bad primes. For the types B_ℓ , C_ℓ , and D_ℓ , $p = 2$, see Wall [42, Theorem 3.7.4]. A very detailed analysis is given for groups of type C_ℓ , D_ℓ when $p = 2$ in [28]. Generating functions for the conjugacy class sizes are given and information on how far they depart from being q -powers are obtained. The type G_2 for $p = 2, 3$ has been analyzed by Enomoto (see [8, Proposition 2.3.(iii), 6.1.(iii)]). For the case F_4 see [33, Theorem 2.1] for $p = 2$ and [35, Theorem 2.1] for $p = 3$. For E_ℓ ($\ell = 6, 7, 8$), $p = 2, 3, 5$ see the papers by Mizuno [26, Lemma 4.3] and [25, Table 9, 10].

4.2. Steinberg groups

${}^2A_\ell$ has no bad primes. For ${}^2D_\ell$, $p = 2$, see Wall [42, Theorem 3.7.4]. For 3D_4 see [6,36]. For 2E_6 see [26, Theorem 6.2].

4.3. Ree–Suzuki groups

For the Suzuki groups 2B_2 and the Ree groups 2G_2 see [9, Table 2.4] and for the Ree groups of type 2F_4 see [34, Table II].

5. Power series and generalized Cayley maps

In this section, we analyze under which hypotheses generalized Cayley maps can be realized by power series. The previous section combined with some other remarks show that this can happen only if p is good. We will assume this is the case in this section. Of course this assumption rules out Ree–Suzuki groups.

Springer proves the existence of an isomorphism of varieties $f : u \rightarrow U$. If we assume that $u \leq \mathfrak{gl}_d(\overline{\mathbb{F}}_p)$, then $f(X) = (f_{ij}(X))$, for d^2 rational functions f_{ij} . On the other hand, a power series depends on at most d coefficients, thus it has fewer chances to exist. We distinguish between two classes of groups.

- (a) Classical: $A_\ell(q)$, $B_\ell(q)$ ($\ell \geq 2$), $C_\ell(q)$ ($\ell \geq 3$), $D_\ell(q)$ ($\ell \geq 4$), ${}^2A_\ell(q)$ ($\ell > 1$), ${}^2D_\ell(q)$ ($\ell \geq 4$);
- (b) Exceptional: $G_2(q)$, $F_4(q)$, $E_\ell(q)$ ($\ell = 6, 7, 8$), ${}^3D_4(q)$, ${}^2E_6(q)$;

5.1. Classical groups

In this case G_F is the central quotient of a suitable subgroup of the full isometry group stabilizing a non-degenerate sesquilinear form E (see [3, Ch. 11, 14]). Since we assume p is good, a maximal unipotent subgroup U_F of G_F is also isomorphic to a p -Sylow subgroup of the full isometry group I (see [18, Ch. 1]). Let $m = 2$ in the unitary case, $m = 1$ otherwise and set $(x_{ij}) = (x_{ij}^q)$, for any $X \in \mathfrak{gl}_d(\mathbb{F}_{q^m})$. Let \mathfrak{g} be the Lie algebra of G . In the unitary case, we may assume that $E = \bar{E}^t$. Then $F^* : X \mapsto -EX^tE^{-1}$ defines an endomorphism of order 2 of \mathfrak{g} . In all cases the Lie algebra of G_F equals

$$\{X \in \mathfrak{gl}_d(\mathbb{F}_{q^m}) : XE + EX^t = 0\},$$

the set of fixed points of F^* . Therefore we will denote it with \mathfrak{g}_{F^*} .

We recall a fact used in [27].

Lemma 5.1. *Let X, Y be nilpotent elements of the full matrix algebra $(K)_d$, K any field, and $f \in K[[x]]$ a power series. Then $XE = EY$ implies $f(X)E = Ef(Y)$.*

Proof. Let $f(x) = \sum_{i \geq 0} f_i x^i$. From the hypothesis it follows that $X^i E = EY^i$, hence $f_i X^i E = E f_i Y^i$ and $f(X)E = Ef(Y)$. \square

Given a maximal nilpotent subalgebra u_{F^*} of \mathfrak{g}_{F^*} , we look for power series f such that:

- (1) $f(u_{F^*}) = U_F$;
- (2) $f(c_{u_{F^*}}(X)) = C_{U_F}(f(X))$, for all $X \in u_{F^*}$.

We call such f a **global Cayley map**. Notice that (1) forces $f_0 = 1$. Moreover, Lemma 1 in [27] shows that for a power series f , (1) implies (2) as long as $f_1 \neq 0$. So if we find a power series f satisfying (1), then the elements of U_F have centralizers of cardinality a power of q , since this holds in u_{F^*} .

Theorem 5.2. *Let G_F be a classical group defined over a field of characteristic p , p a good prime, and $F = \sigma_q \gamma$ a Frobenius endomorphism of G of level q . Then there exists a global Cayley map f such that $f(u_{F^*}) = U_F$. In particular, U_F is a q -power size group.*

Proof. In type A_ℓ , $F = \sigma_q$, set $f(x) = 1 + x$. In the other cases G_F stabilizes a sesquilinear form E . Let $m = 2$ in the unitary case, $m = 1$ otherwise, and set $K = \mathbb{F}_{q^m}$. Choose $\lambda \in K$ such that $\lambda + \lambda^q = 1$. In particular, we may take $\lambda = 1/2$ unless $p = m = 2$. Define

$$f(x) = \frac{1 + \lambda x}{1 - \bar{\lambda} x}.$$

Since $\bar{\bar{\lambda}} = \lambda$, then $f(-\bar{x}) = \overline{f(x)^{-1}}$. If $XE + E\bar{X}^t = 0$, then Lemma 5.1 applied to $Y = -\bar{X}^t$ implies $f(X)E\overline{f(X)^t} = E$. If $G \leq GL_d(K)$, let $u_d(K)$ be the subalgebra of lower triangular matrices in $\mathfrak{gl}_d(K)$. Since u_{F^*} contains only nilpotent endomorphisms of K^d , then u_{F^*} is conjugate in $GL_d(K)$ to a Lie subalgebra of $u_d(K)$ (see [10, Proposition 2.1.4]). Therefore we may assume that E is chosen so that $u_{F^*} \leq u_d(K)$. Let $U_d(K)$ be the subgroup of lower unitriangular matrices, then

$$f(u_{F^*}) \subseteq U_d(K) \cap G_F.$$

Since $f(u_d(K)) = U_d(K)$, any $x \in U_d(K) \cap G_F$ has shape $f(X)$, for some $X \in u_d(K)$. Set $Z = f(X) - I$ and $Y = f(-\bar{X}^t) - I$, then $ZE = EY$. Since $f_1 = \lambda + \bar{\lambda} = 1$, $f - 1$ has an inverse g as power series and we obtain

$$XE = g(Z)E = Eg(Y) = -E\bar{X}^t$$

and $X \in u_{F^*}$. Thus $f(u_{F^*}) = U_d(K) \cap G_F$ is a p -Sylow subgroup U_F of G_F . \square

5.2. Exceptional groups

Fix a Cartan subalgebra \mathfrak{h} of $\mathfrak{g}_{\mathbb{C}}$ and let Φ denote the associated root system, Δ a fundamental system of Φ . A **Chevalley basis** h_δ ($\delta \in \Delta$), e_α ($\alpha \in \Phi$) of $\mathfrak{g}(\mathbb{C})$ allows one to define a \mathbb{Z} -Lie ring $\mathfrak{g}(\mathbb{Z})$. Given any field K , we obtain a K -Lie algebra $\mathfrak{g}(K) = \mathfrak{g}(\mathbb{Z}) \otimes_{\mathbb{Z}} K$. Since $\text{ad } e_\alpha$ is a nilpotent endomorphism of $\mathfrak{g}(\mathbb{C})$, $x_\alpha(t) = \exp(\text{ad } t e_\alpha)$ makes sense for any $t \in \mathbb{C}$. It turns out that $x_\alpha(t) \in (\mathbb{Z}[t])_{\dim \mathfrak{g}}$, hence it makes sense when t varies in any field K . As is well-known $G(K) = \langle x_\alpha(t) : \alpha \in \Phi, t \in K \rangle$ is called the **(adjoint) Chevalley group** associated to $\mathfrak{g}(K)$.

For any representation ϕ of $\mathfrak{g}(\mathbb{C})$, a theorem of Kostant allows us to define in a similar way groups $G_\phi(K)$ as subgroups of $GL_d(K)$, where d denotes the degree of ϕ . Moreover, G_ϕ admits $G = G_{\text{ad}}$, the usual Chevalley group, as a central quotient and the kernel has order 1, 2, 3 or 4. Since these values are coprime to p whenever p is good, $U = U_{\text{ad}} \simeq U_\phi$. So we may identify U with U_ϕ and drop ϕ in the notation. Hence from now on we assume that $\mathfrak{g}(K) \leq \mathfrak{gl}_d(K)$ or, equivalently, that ϕ is the identity map. Then $U = \langle \exp(t e_\alpha) : t \in K, \alpha \succ 0 \rangle$, where $\exp(t e_\alpha)$ is defined in the usual way when $p \geq d$, otherwise its definition requires a more subtle procedure (see [12, Ch. VII] and [27, Section 3]).

Unfortunately such a procedure does not allow one to define $\exp(e)$ for an arbitrary element $e \in \mathfrak{u}$. Let ν be the smallest positive integer k such that $e^k = 0$, usually ν is called the **nil index** of e , then $\exp(e)$ is defined if $p \geq \nu$. Otherwise there might exist a \mathbb{Z} -form invariant under $e_\alpha^i/i!$, $\alpha \succ 0$, but not under $\exp(e)$. An example is given by the Lie algebra of type A_2 and $e = e_1 + e_2$ (see [41, pag. 40]).

In order to build global Cayley maps for Lie algebras, it is therefore convenient to deal with elements of low nil index. Since the nil index of $\phi(y)$, y a nilpotent element, increases with the degree d of ϕ , we will consider faithful representations ϕ of minimal degree. In the exceptional cases, they are the **standard** and the **adjoint** ones. We refer to them as st and ad .

Let $\mathfrak{g} = \mathfrak{g}(\overline{\mathbb{F}}_p)$, γ a graph automorphism of \mathfrak{g} of order m and q a power of p . Then $F^*(te_\alpha) = t^q \gamma(e_\alpha)$ defines an endomorphism of \mathfrak{g} . Notice that $\mathfrak{u} = \langle te_\alpha : t \in \overline{\mathbb{F}}_p, \alpha \succ 0 \rangle$ is stable under F^* . Given $\exp(t \text{ ad } e_\alpha) \in U_{\text{ad}}(\overline{\mathbb{F}}_p)$, considered as an automorphism of \mathfrak{g} , we have

$$F^* \exp(t \text{ ad } e_\alpha) (F^*)^{-1} = \exp(F^*(t \text{ ad } e_\alpha)) = \exp(t^q \text{ ad } \gamma(e_\alpha)) \in U_{\text{ad}}(\overline{\mathbb{F}}_p). \tag{5.1}$$

Thus conjugation by F^* induces an endomorphism F of $U_{\text{ad}}(\overline{\mathbb{F}}_p)$. Using the isomorphism between $U_{\text{ad}}(\overline{\mathbb{F}}_p)$ and U we also obtain an endomorphism of U which we still denote by F . We say that F^* induces F on U .

Given the Lie algebra $\mathfrak{u} = \langle te_\alpha : t \in \overline{\mathbb{F}}_p, \alpha > 0 \rangle$, the unipotent group $U = \langle \exp(te_\alpha) : t \in \overline{\mathbb{F}}_p, \alpha > 0 \rangle$, F^* an endomorphism of \mathfrak{u} inducing a Frobenius endomorphism F of U , let u_{F^*}, U_F be the corresponding set of fixed points. As in the classical case we look for power series f such that

$$f(u_{F^*}) = U_F. \tag{5.2}$$

Since $|u_{F^*}| = |U_F|$, f must induce a bijection, thus $f_1 \neq 0$, where $f(x) = 1 + \sum_{i \geq 1} f_i x^i$. Thus we may assume that $f_1 = 1$, so $f - 1$ is an invertible element of $K[[x]]$ with respect to composition. Let g be the inverse to $f - 1$ that is, $g(f(x) - 1) = x$, then

$$X * Y := g(f(X)f(Y) - 1) \tag{5.3}$$

defines a group structure on u_{F^*} . In particular, u_{F^*} must be closed under $*$.

We prove that for a group of exceptional type a power series f satisfying Eqs. (5.2) and (5.3) exists only if $p \geq c$, where c denotes the nilpotency class of u_{F^*} and, in this case, f must be a (truncated) exponential, except for type 3D_4 and G_2 . For these types the following power series

$$\wp(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{144}$$

works for $p \geq 5$.

This improves Theorem 6 in [27] where we showed that the exponential map is defined on \mathfrak{u} whenever $p \geq \nu$, ν the nil index of the associative algebra generated by \mathfrak{u} . We use the theory of **regular** p -groups to reduce this bound to the nilpotency class of \mathfrak{u} .

5.3. Twisted Lie subalgebras

We first provide a more detailed description of u_{F^*} . We start by pointing out the structure of a basis for u_{F^*} , when $F^* = \gamma\sigma_q$ is a Frobenius endomorphism of \mathfrak{g} , γ a graph automorphism of order m . For simplicity we assume that the underlying field is $K = \mathbb{F}_{q^m}$. We recall that given a finite Galois extension K/L with Galois group Σ , we say that $\omega \in K$ is **normal** over L if $\{\omega^\sigma : \sigma \in \Sigma\}$ is an L -basis for K (see Section 3).

Lemma 5.3. *Let γ be a graph automorphism of order m , $K = \mathbb{F}_{q^m}$, $L = \mathbb{F}_q$, and ω a normal element for K/L . Then u_{F^*} admits as L -basis the elements*

$$\widehat{e_\alpha(\omega)} := \sum_{i=0}^{m-1} \omega^{q^i} \gamma^i(e_\alpha),$$

where $\alpha > 0$.

Proof. Notice that $\widehat{e_\alpha(\omega)}$ is F^* -invariant. Conversely, if $x = \sum_{\alpha > 0} \lambda_\alpha e_\alpha$ is F^* -invariant, then $\lambda_{\gamma(\alpha)} = \lambda_\alpha^q$, and x is the sum of terms of shape $\widehat{e_\alpha(\lambda_\alpha)}$. Since different orbits involve disjoint subset of vectors e_α , we only need to prove that $\widehat{e_\alpha(\omega)}$ are L -linearly independent when α varies in a γ -orbit. Let s denote the size of such orbit. Then $m = st$, for some integer t . Set $\tau = \sum_{i=0}^{t-1} \omega^{q^i}$. Namely τ is the field trace from K to $J = \mathbb{F}_{q^s}$ of ω . Since $\tau^{q^i}, 0 \leq i \leq s - 1$, have disjoint supports with respect to the basis $\{\omega^{q^i} : 0 \leq i \leq m - 1\}$, ω normal for K/L forces τ normal for J/L . Now the transition matrix from the elements $e_{\gamma^i(\alpha)}$ to $\widehat{e_\alpha(\omega)}$ is $T = (\eta_i \tau^{q^{i-j}})$, where $\gamma^i(e_\alpha) = \eta_i e_{\gamma^i(\alpha)}$, $\eta_i = \pm 1$. Let $T_0 = (\tau^{q^{i-j}})$, then $\det T = \pm \det T_0$. Since τ is normal for J/L , $\det T_0 \neq 0$ (see [21, Corollary 2.38]). \square

As a consequence $\mathfrak{u}(K) \simeq u_{F^*} \otimes_L K$ and, similarly, $\mathfrak{u}^-(K) \simeq (u^-)_{F^*} \otimes_L K$. Therefore $\mathfrak{g}(K)_{F^*} \otimes K \simeq \mathfrak{g}(K)$.

Given an integer s , we define iteratively left-normed commutators of **weight** s in \mathfrak{u} . If $s = 1$, they are elements of \mathfrak{u} , otherwise they are elements of shape $[x, y]$, where x is a left-normed commutator of weight $s - 1$ and y of weight 1. We recall that Jacobi's identity implies that any commutator is a linear combination of left-normed ones.

Lemma 5.4. *Let h denote the **Coxeter number** of the root system Φ . Then h has four different descriptions:*

- (1) the order of the longest element in the Weyl group \mathcal{W} of Φ ;
- (2) $\text{ht}(\rho) + 1$, where ρ is the highest root in Φ^+
- (3) $|\Phi|/\ell$, where ℓ is the rank of Φ ;
- (4) $c + 1$, where c denotes the nilpotency class of \mathfrak{u} , when $p > 3$.

Proof. The equivalence of (1)–(3) is proved in [13, 3.18] and [2, Théorème 1, Ch. V, Section 6.2]. Now $\mathfrak{u} = \bigoplus_{\alpha > 0} \mathfrak{g}_\alpha$ and

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta},$$

this last term being zero if $\alpha + \beta \notin \Phi$. So any commutator of weight h among elements of \mathfrak{u} annihilates because of (2). On the other hand, there exists a sequence (i_1, \dots, i_{h-1}) , $1 \leq i_j \leq \ell$ such that $\sum_{j=1}^k \alpha_{i_j} \in \Phi^+$, for any $1 \leq k \leq h - 1$ and

$$\sum_{j=1}^k \alpha_{i_j} = \rho$$

(see [3, Lemma 3.6.2]). So the left-normed commutator

$$[e_{\alpha_{i_1}}, \dots, e_{\alpha_{i_{h-1}}}]$$

equals μe_ρ , for some $\mu \in \mathbb{F}_p$. Set $[e_\alpha, e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta}$, then $N_{\alpha,\beta} \in \{\pm j : 0 \leq j \leq i\}$, according as α, β generate a rank 2 system of type Φ_i , where Φ_i equals $A_1 \times A_1, A_2, B_2$ or G_2 , for $0 \leq i \leq 3$. Thus if $p > 3$, then $\mu \neq 0$ and h equals the nilpotency class of \mathfrak{u} . \square

When Φ has type B_ℓ, C_ℓ or F_4 and $p = 2$ then $c = 2(\ell - 1), 2\ell - 1$ and 9 , respectively. Otherwise $c = h$. If Φ has type $G_2, p = 2, 3$, then $c = 4$.

We extend this result even to the twisted case, namely we prove that the nilpotency class of \mathfrak{u}_{F^*} and \mathfrak{u} are equal.

Lemma 5.5. *The nilpotency class of \mathfrak{u}_{F^*} and \mathfrak{u} are equal.*

Proof. We only need consider the case $\gamma \neq id$. Since $\mathfrak{u}_{F^*} \leq \mathfrak{u}$, $c(\mathfrak{u}_{F^*}) \leq c(\mathfrak{u})$. Let c be the former value, then

$$[x_1, \dots, x_{c+1}] = 0,$$

for all $x_i \in \mathfrak{u}_{F^*}$. Without loss of generality, we may assume that x_1 has shape $\widehat{e_\alpha(\beta)}$. Since e_α is a K -linear combination of $\widehat{e_\alpha(\beta)}, \beta \in K$, then

$$[e_\alpha, x_2, \dots, x_{c+1}] = 0.$$

In a similar way we may substitute any x_i with generators of \mathfrak{u} , so $c \geq c(\mathfrak{u})$. \square

5.4. Baker–Campbell–Hausdorff Formula in positive characteristic

We now exploit this result to prove that if $p \geq h$ and $X \in \mathfrak{u}$, then $X^p = 0$ (this is an alternative proof of Corollary 4.4 using Remark 4.4.(1) in [24]). Let K be a field of characteristic p , A the free algebra of rank 2 over K with indeterminates x, y . Then

$$(x + y)^p - x^p - y^p = \sum_{i=1}^{p-1} s_i(x, y),$$

where $s_i(x, y)$ is a nested commutator of weight p (see [10, 1.13]). Now $X = \sum_{\alpha \in \Phi^+} a_\alpha e_\alpha$ and, by induction on $|\Phi^+|$, we deduce that

$$X^p = \sum_{\alpha \in \Phi^+} a_\alpha^p e_\alpha^p + Y,$$

where Y is a linear combination of commutators of weight p . Since $p \geq h > \text{ht}(\rho)$, $Y = 0$. If $p > 3$, then $e_\alpha^p = 0$, so $X^p = 0$. We recall that we consider e_α as an element of $(K)_d$, d the degree of the standard or the adjoint representation of \mathfrak{g} . In this case the nil index of e_α is at most 3 in the standard case and at most 4 in the adjoint case and when α is short in a system of type G_2 . Therefore \exp is defined on all of \mathfrak{u} . Let $A = \mathbb{Z}[t_\alpha, s_\alpha : \alpha \in \Phi^+]$, the polynomial ring over \mathbb{Z} of rank $2|\Phi^+|$. Let ϕ be the standard or adjoint representation for \mathfrak{g} and set $X = \sum_{\alpha \in \Phi^+} t_\alpha \phi(e_\alpha)$. Then $X^p \in (pA)_d$, d the degree of ϕ , and $X^{(p)} := X^p/p \in (A)_d$. If $\phi = \text{ad}$, then $X^{2h-1} = 0$ (see [4, Proposition 5.5.2]). A careful analysis of the quoted proposition yields that $2h - 1$ is the nil index of X . When $\phi = \text{st}$, the standard representation, we exploit MAGMA to determine the nil index ν_{st} of X . Since ν_{st} is unaffected by roots of height greater than 1, we may assume $X = \sum_{\alpha \in \Delta} t_\alpha \text{st}(e_\alpha)$ reducing the complexity of the calculation. Otherwise we may obtain ν_{st} as the diameter of a digraph built from a module of minimal dimension for \mathfrak{g} (see [27, Section 3]):

G	3D_4	G_2	F_4	E_6	2E_6	E_7	E_8
h	6	6	12	12	12	18	30
ν_{st}	7	7	17	17	17	28	59
d_{st}	7	8	26	27	27	56	248
ν_{ad}	11	11	23	23	23	35	59
d_{ad}	14	28	52	78	78	133	248

In particular $\nu_{st} \leq \nu_{ad} = 2h - 1 < 2p$, for $p \geq h$. Set $c_i = (p + i)!/p$, for $0 \leq i \leq p - 1$, $s = (p - 1)!$ and $B = \mathbb{Z}[\frac{1}{s}, t_\alpha, s_\alpha : \alpha \in \Phi^+]$, the polynomial ring over $\mathbb{Z}[\frac{1}{s}]$ of rank $|\Phi|$. Then the formal exponential of X

$$\varepsilon(X) := \sum_{i=0}^{p-1} X^i/i! + X^{(p)} \sum_{i=0}^{p-1} X^i/c_i,$$

belongs to $(B)_d$. Notice that the definition of ε is already used by Chevalley to build the elements $x_\alpha(t)$ in small characteristic (see [3, Section 4.3]). Set $Y = \sum_{\alpha > 0} s_\alpha \phi(e_\alpha)$.

We recall that in the formal non-commutative power series ring $\mathbb{Q}\langle x, y \rangle$

$$\exp(x) \exp(y) = \exp(H(x, y)),$$

where $H(x, y) = \sum_{n \geq 1} H_n(x, y)$ is the so-called **Baker–Campbell–Hausdorff series**, a Lie series in the non-commuting variables x, y . Given $n \in \mathbb{N}$, then the homogeneous component $H_n(x, y)$ of degree n of $H(x, y)$ is a Lie polynomial in $\mathbb{Z}[\frac{1}{n!}]\langle x, y \rangle$ (see [16, pag. 173]).

We now specialize x, y to X, Y , so

$$\varepsilon(X)\varepsilon(Y) = \varepsilon(Z)$$

where $Z = H(X, Y) = \sum_{n \geq 1} H_n(X, Y)$. Since $p \geq h$, $H_n(X, Y) = 0$ for $n \geq p$. So

$$Z \in (B)_d.$$

Specializing the variables t_α, s_α to elements of K , we define an exponential-like map on \mathfrak{u} realizing an isomorphism with U via ε . Moreover, if we consider $\phi = \text{ad}$, then $\varepsilon(F^*(X)) = F(\varepsilon(X))$, so

$$\varepsilon(u_{F^*}) = U_F.$$

We have therefore proved:

Theorem 5.6. *Let $U = \langle \varepsilon(t \text{ ad } e_\alpha) | \alpha \in \Phi^+, t \in K \rangle$, Φ^+ the positive roots of an indecomposable root system, K a field of characteristic p . If $p \geq h$, then ε is a global Cayley map for \mathfrak{u} . In particular, if F is a Frobenius endomorphism of G of level q , then U_F is a q -power size group.*

Since $\varepsilon(X)^p = \varepsilon(pX) = I$, we obtain that U has exponent p , when $p \geq h$. Hence the logarithmic map is defined on U , too. We would like to give an alternative proof avoiding the use of the Baker–Campbell–Hausdorff Formula.

Given a p -group P , let

$$\Omega_i(P) = \langle u \in P | u^{p^i} = 1 \rangle.$$

We recall that a p -group P is called **regular** if for any $a, b \in P$

$$(ab)^p = a^p b^p \prod_i c_i^p,$$

for some $c_i \in \langle a, b \rangle$. In a regular p -group P , $\Omega_i(P) = \{u \in P | u^{p^i} = 1\}$ (see [15, III.10]).

Now $U = \langle \varepsilon(te_\alpha) | \alpha \in \Phi^+, t \in K \rangle$ is a nilpotent group with nilpotency class at most h . In fact, the i th term of the lower central series of U is contained in $U_i = \langle \varepsilon(te_\alpha) | \text{ht}(\alpha) \geq i, t \in K \rangle$. They actually coincide whenever the characteristic p of K is **nonsingular**, that is, greater than the number of bonds in the Dynkin diagram for G (see [3, Section 5.3], [9, Section 1.12]).

Theorem 5.7. *Let $U = \langle \varepsilon(te_\alpha) | \alpha \in \Phi^+, t \in K \rangle$, Φ^+ the positive roots of an indecomposable root system, h its Coxeter number, K a field of characteristic $p \geq h$, F a Frobenius endomorphism of U . Then U_F has exponent p .*

Proof. Assume $p \geq h$. Then p is greater than the nilpotency class of U . By [15, Satz III.10.2.a)], U is a regular p -group. Now U_F is generated by $\prod_i \exp(F^{*i}(te_\alpha))$, where the product ranges over γ -orbits, γ the graph automorphism occurring in F^* . We claim that the Chevalley's basis elements in a given orbit either commute or the orbit generates a system of type A_2 . This is obvious when the orbit has size 1. Otherwise only types ${}^3D_4, {}^2E_6$ and ${}^2A_\ell, \ell > 1$ occur. The first two can be checked directly. In the last case a typical element u of the corresponding root subgroup has shape

$$x_\alpha(a)x_\beta(b)x_{\alpha+\beta}(c)$$

with some relations among a, b, c (see [9, Table 2.4.IV]) and $x_\mu(t)$ denotes $\varepsilon(te_\mu)$, $t \in K, \mu \in \Phi^+$. Then

$$u^p = x_\alpha(pa)x_\beta(pb)x_{\alpha+\beta}(pc + \binom{p}{2}ab) = 1.$$

Therefore U_F is generated by elements of order p . Thus $U_F = \Omega_1(U_F) = \{x \in U_F | x^p = 1\}$ and $\text{Exp}(U_F) = p$. \square

This is actually a particular case of the Order Formula by Testerman (see [41, Section 2]).

We finally point out that $X^p = 0, \forall X \in \mathfrak{u}$ and $u^p = 1, \forall u \in U$, namely that the (truncated) exponential and logarithmic map are defined on \mathfrak{u} and U , respectively, are not sufficient conditions to assure that $H(X, Y) \in \mathfrak{u}, \forall X, Y \in \mathfrak{u}$.

Example 5.8. *Let $K = \mathbb{F}_2, L$ the Lie subalgebra over K generated by $x = e_{12} \otimes I_2$ and $y = I_2 \otimes e_{12}$, where e_{12} is the elementary 2×2 -matrix and \otimes denotes the Kronecker product. Then $\dim L = 2, xy = yx, L$ is commutative and $t^2 = 0$, for any $t \in L$. So \exp is defined on all of L . But $\log(\exp(x)\exp(y)) = x + y + xy \notin L$. Notice that $U = \langle \exp(t) : t \in L \rangle$ is an abelian unipotent subgroup of $\text{GL}_4(2)$ of exponent 2 and order 8.*

5.5. Triality and G_2

We prove that the previous bound may be reduced further when G has type 3D_4 or G_2 . In these cases a slight modification

$$\wp(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{144}$$

of the exponential map defines a global Cayley map for $p \geq 5$.

The general scheme is to determine the maximum nil index of elements in the associative algebra generated by $\phi(u_{F^*})$. This turns out to be 7. We try first to find good candidates in specific examples and then prove they work whenever $p \geq 5$.

For the triality group 3D_4 , we consider a cyclic Galois extension of degree 3, for example $\mathbb{Q}(a)/\mathbb{Q}$, where $a^3 - a^2 - 2a + 1 = 0$. Let σ denote a generator for the associated Galois group and define

$$F^*(te_\alpha) = t^\sigma \gamma(e_\alpha),$$

γ a graph automorphism of order 3. Lemma 5.3 shows that $\widehat{e_\alpha(a)}, \alpha > 0$, constitute a basis for u_{F^*} . We build the polynomial ring over $\mathbb{Q}(a)$ of rank 5 and the power series ring $R = \mathbb{Q}(a)[f_2, \dots, f_6][[x]]$. Set

$$f = 1 + x + \sum_{i=2}^6 f_i x^i.$$

Then $f - 1$ admits an inverse power series g , i.e. $g(f(x) - 1) = x$. Now we pick 24 random integers $x_\alpha, y_\alpha, \alpha > 0$, set $X = \sum_{\alpha > 0} x_\alpha \phi(\widehat{e_\alpha(a)})$, $Y = \sum_{\alpha > 0} y_\alpha \phi(\widehat{e_\alpha(a)})$, calculate

$$g(f(X)f(Y) - 1),$$

and impose that this matrix belong to the \mathbb{Q} -linear span of $\phi(\widehat{e_\alpha(a)})$. This is equivalent to a polynomial system in the f_i 's indeterminates. A Groebner's basis calculation in MAGMA yields $f_i = 1/i!$, $2 \leq i \leq 4$ and $f_5 - f_6 - 1/144 = 0$. The same result is obtained considering a (cyclic) Galois extension of degree 3 over finite fields of characteristic at least 5. We simplify the shape of our candidate power series setting $f_5 = 1/144$ and $f_6 = 0$ and set

$$\wp(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{144}.$$

Once we have a candidate for a global Cayley map we check that this really works for any extension and independently from random choices. So we consider $A = \mathbb{Z}[\frac{1}{6}, a_i, x_\alpha, y_\alpha], i \in \mathbb{Z}/3, \alpha > 0$, the polynomial ring of rank 27 over $\mathbb{Z}[\frac{1}{6}]$. We define σ as the automorphism of A exchanging the a_i 's cyclically and fixing the other variables. Set $\widehat{e_\alpha(a)} = \sum_{i=0}^2 a_i \gamma^i(e_\alpha)$, $X = \sum_{\alpha > 0} x_\alpha \text{st}(\widehat{e_\alpha(a)})$, $Y = \sum_{\alpha > 0} y_\alpha \text{st}(\widehat{e_\alpha(a)})$. We calculate the Baker–Campbell–Hausdorff series up to weight 4 in X, Y , $H^{(4)}(X, Y)$. Then

$$\wp^{-1}(\wp(x)\wp(y) - 1) - H^{(4)}(X, Y)$$

turns out to belong to $A\phi(\widehat{e_\rho(a)})$, where ρ denotes the highest root.

A similar, but simpler, approach works for G_2 and yields the same conclusions.

Theorem 5.9. Let u_{F^*} be of type G_2 or 3D_4 over a field K of characteristic $p \geq 5$. Then $\wp(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{144}$ defines a global Cayley map between u_{F^*} and U_F . In particular U_F is a q -power-size group, where q is the level of F .

5.6. Final result

With the support of MAGMA, we have checked which invertible power series f endow u with a group structure and commute with the action of the Frobenius endomorphism F .

Theorem 5.10. Let G be a simple linear algebraic group of exceptional type, F a Frobenius endomorphism of G of level $q = p^a$, $p, a \in \mathbb{N}, p$ prime, induced by F^* . Let U_F be a maximal unipotent subgroup of G_F . Then $U_F = f(u_{F^*})$ for a global Cayley map f iff $p \geq b(G)$, where $b(G)$ depends only on the type of G and

G	3D_4	G_2	F_4	E_6	2E_6	E_7	E_8
$b(G)$	5	5	13	13	13	19	31

Moreover, f is an exponential map except when G has type 3D_4 or G_2 .

Proof. A few comments are in order to explain how MAGMA was used to deal with (5.3). In the untwisted case set $K = \mathbb{Q}$, otherwise let m denote the order of the graph automorphism occurring in F and set $K = \mathbb{Q}(a_0, \dots, a_{m-1})$. In the latter case we consider an automorphism σ of L defined via $a_i^\sigma = a_{i+1}$ where the indices are considered modulo m . Then we proceed

exactly as in Section 5.5, namely we set $X = \sum_{\alpha > 0} x_\alpha \widehat{\text{st}(e_\alpha(a))}$, $Y = \sum_{\alpha > 0} y_\alpha \widehat{\text{st}(e_\alpha(a))}$, where $a = a_0$ and x_α, y_α are randomly chosen integers. We let $R = K(f_2, \dots, f_\nu)[[x]]$, where $\nu = \nu_{\text{st}}$ is the nil index of the associative hull of u in the standard representation and $f = 1 + x + \sum_{i=2}^\nu f_i x^i$. Build $X * Y = g(f(X)f(Y) - 1)$, where g is the inverse to $f - 1$. Forcing $X * Y$ in u_{F^*} imposes polynomial conditions on the f_i 's. Using a Groebner basis calculation in MAGMA, we obtain that f is an exponential map or \wp when G has type 3D_4 or G_2 . This approach is feasible unless G has type E_8 , in which case a more sophisticated approach is needed. Namely, we consider the Lie algebra of strictly upper triangular matrices of degree d over K , $u_d(K)$. Fix $k \geq 1$ and work modulo $u_d(K)^k$ obtaining the coefficients of f_i as i increases.

As we have seen, this does not rule out some strange behavior in small characteristic. We need of course consider only primes less than the Coxeter number. Set $K = \mathbb{F}_{p^m}$, where $p < h$, and $A = K[x_\alpha]$ the polynomial ring of rank $|\Phi^+|$, $X = \sum_{\alpha > 0} x_\alpha \widehat{\text{st}(e_\alpha(a))}$. Then again MAGMA shows that $X^p \notin (pA)_d$. Therefore we consider only power series truncated at the p th term. An analysis similar to the previous one with the prime subfield \mathbb{Q} substituted with \mathbb{F}_p shows that no global Cayley map exists in these cases. \square

5.7. Closure under scalars

We prove that truncated logarithmic and exponential maps show up even if we relax the above conditions. One might wonder whether the Lie algebra of $U_F \leq \text{GL}_d(K)$ is really the structure one needs to obtain information the conjugacy class sizes of U_F . Since our data is the nilpotent group U_F we look for a power series $g(x) = x + \sum_{j>1} g_j x^j \in K[[x]]$ such that $\mathcal{U}_d(K) := g(U_F - 1)$ is a K -vector space. Notice that if such a power series exists, then class sizes of U_F are $|K|$ -powers. In fact, Lemma 1 in [27] shows that g preserves centralizers and centralizers in $\mathcal{U}_d(K)$ are obviously K -subspaces.

Denote with f the inverse to g with respect to composition and set $u_\lambda := 1 + f(\lambda g(u - 1))$, we must have $u_\lambda \in U_F$ $u \in U_F, \lambda \in K$. This element generalizes the one defined by the process of saturation when one takes g as the logarithmic map (see [30] and [31, Section 4]).

It turns out that for G of exceptional type, $u \in U_F$ regular unipotent, $u_2 \in U_F$ iff g is a truncated logarithm, except in type G_2 and 3D_4 , as pointed out in the previous subsection.

We establish an analogous result to Lemma 5.3 for groups. In order to control the generators of U_F , we need information on elements e_α belonging to the same γ -orbit.

Lemma 5.11. *Let \mathfrak{g} be an exceptional Lie algebra. Then $[e_\alpha, e_\beta] = 0$ when α, β belong to the same γ -orbit.*

Proof. If $m = 1$, there is nothing to prove. If $m = 2$, then \mathfrak{g} has type E_6 and every orbit has type A_1 or $A_1 \times A_1$. If $m = 3$, then \mathfrak{g} has type D_4 and every orbit has type A_1 or $A_1 \times A_1 \times A_1$. \square

It follows that U_F is generated by $x_\alpha(t), t \in K$ in case A_1 , $x_\alpha(t)x_{\gamma(\alpha)}(t^q)$ in case $A_1 \times A_1$ and $\prod_{i=0}^2 x_{\gamma^i(\alpha)}(t^{q^i})$ in case $A_1 \times A_1 \times A_1$ (see [9, Theorem 2.3.7, Section 2.4]).

Theorem 5.12. *Let G be an exceptional group of Lie type, K a finite field of characteristic $p \geq \nu_{\text{st}}$, the nil index of the standard representation for G , F a Frobenius automorphism of G . Then either the only power series $g \in K[[x]]$ such that $\mathcal{U}_d := g(U_F - 1)$ has a K -vector space structure is a truncated logarithm or G has type G_2 or 3D_4 and $g \equiv_{x^5} (\wp - 1)^{-1}$.*

Proof. Let $F = \sigma_q \gamma$, γ of order m and $K = \mathbb{F}_{q^m}$. Set $L = \mathbb{F}_q$ and let ρ be a normal element for K/L . Then any $\tau \in K$ has shape $\sum_{i=0}^{m-1} \lambda_j \rho^{q^i}, \lambda_j \in L$. Now U_F is generated by $\widehat{x_\alpha(\tau)} := \prod_{i=0}^{s-1} x_{\gamma^i(\alpha)}(\tau^{q^i})$, where s is the size of the γ -orbit of $\alpha \in \Phi^+$ and $\tau^{q^s} = \tau$.

By Lemma 5.11, $\widehat{x_\alpha(\tau)} = \prod_j x_\alpha(\lambda_j \rho^{q^j}) = \prod_j x_{\gamma^{-j}(\alpha)}(\lambda_j \rho)$. Now $m = 1, 2, 3$, so either $s = m$ or $s = 1$. If $s = 1$, then $\tau \in L$. In both cases the number of λ 's occurring equals s . So we may express the latter element as $\prod_\beta x_\beta(\widehat{\lambda_\beta \rho})$, where β ranges in the γ -orbit of α . We now consider the ring $A = \mathbb{Z}_{(p)}[a_i, s_\alpha], i \in \mathbb{Z}/m$ and define $\sigma \in \text{Aut}(R)$ as $a_i^\sigma = a_{i+1}, s_\alpha^\sigma = s_\alpha$. We define by analogy $\widehat{x_\alpha(s_\alpha a_0)}$ as $\prod_j x_{\gamma^j(\alpha)}(s_\alpha a_j)$ and consider the element $r = \prod_i \prod_{\alpha \in O_j} \widehat{x_\alpha(s_\alpha a_0)}$, where the O_j 's are the γ -orbits. If we specialize a_i to ρ^{q^i} and s_α to λ_α , we obtain an element in U_F . Consider now $u = \prod_\delta \widehat{x_\delta(1)}$, where δ varies over a set of representatives of γ -orbits in the fundamental system Δ . We apply to r and u the standard representation of $G(A)$. Set $n = \text{st}(u) - 1 \in (A)_d$, where d is the degree of st . We consider the power series ring $P = A[g_j : 2 \leq j < \nu_{\text{st}}][[x]]$ and set $g = x + \sum_j g_j x^j$. We determine the inverse f to g and set $r_2 = 1 + f(2g(n))$. The condition $r_2 = u$ is equivalent to a polynomial system in s_α, g_j . Let I be the associated ideal. Using MAGMA we calculate the elimination ideal in $A[g_j : 2 \leq j < \nu_{\text{st}}]$ with respect to the g_j 's variables, i.e. $I \cap \mathbb{Z}_{(p)}[g_j]$. In all cases we obtain $g_j = \frac{(-1)^{j+1}}{j}$ except when G has type G_2 or 3D_4 . In these cases $g_j = \frac{(-1)^{j+1}}{j}, j = 2, 3, 4$ and $g_5 = -\frac{2}{5}g_6 + \frac{2}{15}$. Setting $g_6 = -\frac{49}{720}$ we get $g = -\sum_{j=1}^4 \frac{(-x)^j}{j} + \frac{29}{144}x^5 - \frac{49}{288}x^6$ and $f = \wp - 1$. So they are both defined for $p \geq 5$. \square

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