Def $K \geq F$ finita si dice radicale se $\exists\, a \in K$

i) $K = F[a]$

ii) $\exists\, n \in \mathbb{N}$ tale che $a^n = b \in F$

ossia (in modo ambiguo) $K = F[\sqrt[n]{b}]$

Def $F = K_0 \leq K_1 \ldots \leq K_m = K$ $K \geq F$ si dice radicale iterata se $\forall i$ $K_i \leq K_{i+1}$ est. rad.

Def Dato $f(x) \in F[x]$ si dice risolvibile per radicali se $\exists\, L \geq F$ radicale iterata tale $L \geq Sp_F(f(x))$

Es: $f(x) = x^2 + ax + b \in F[x]$ char $F \neq 2$

$K = F[\sqrt{\Delta}]$ $\Delta = a^2 - 4b$

$\rho_{1,2} = \dfrac{-a \pm \sqrt{\Delta}}{2}$

$n \in \mathbb{N}$ $x^n - 1 \in \mathbb{Q}[x]$ $\Phi_m(x) = \prod\limits_{j \in \mathbb{Z}/_m^*} (x - \omega^j)$ $\operatorname{ord}(\omega) = m$

$K = \mathbb{Q}(\omega) \geq \mathbb{Q}$ è Galois $|K : \mathbb{Q}| = \varphi(n)$

$Gal(K/\mathbb{Q}) \simeq \mathbb{Z}/_m^*$ è abeliano (caso particolare di gruppi risolubile).

Es: $n = 3$ $\operatorname{ord}(\omega) = 3$ $K = \mathbb{Q}(\omega)$ $|K : \mathbb{Q}| = 2$

$\min_{\mathbb{Q}}(\omega) = x^2 + x + 1 = \phi_3(x)$ $\omega = \sqrt[3]{1}$ $1 \neq \omega$

$\omega \in \mathbb{C}$ $Re(\omega), Im(\omega)$? $K \geq \mathbb{Q}$ estensione radicale

$\exists\, m \in \mathbb{N}$ $a \in K$: i) $K = \mathbb{Q}[a]$. $K = \mathbb{Q}[\omega]$

ii) $a^m = b \in \mathbb{Q}$ $\omega^3 = 1$

$m = \varphi(n) = 2$ a si ottiene modificando ω.

$\uparrow$

$\lambda = -1$ $\operatorname{ord}(\lambda) = \varphi(n) = 2$

$\boxed{a = \omega + \lambda\, \sigma(\omega)}$ dove $\langle \sigma \rangle = Gal(K/\mathbb{Q})$

$\mathbb{Z}/_3^* = \langle 2 \rangle$ $\sigma(\omega) = \omega^2$

$\sigma(a) = \omega^2 + \lambda\omega = \lambda a$, infatti $\lambda a = \lambda\omega + \omega^2$

Algebra Lineare $\sigma \in C_A(F')$ σ è F-lineare

$\sigma(a) = \lambda a$ $\quad a \in K$ autovettore di autovalore λ rispetto a σ

$\sigma(a^2) = \sigma(a)^2 = (\lambda a)^2 = \lambda^2 a^2 = a^2 \in C_K(\langle\sigma\rangle) = \mathbb{Q}$

$a^2 = b \in \mathbb{Q} \quad a \notin \mathbb{Q} \quad K = \mathbb{Q}[a],\ a^2 = b$

$\omega^2 + \omega + 1 = 0$; $\quad a = \omega - \omega^2 \quad a^2 = \omega^2 + \omega^4 - 2\omega^3 = \omega^2 + \omega - 2$

ossia $a^2 = \underbrace{\omega^2 + \omega + 1} - 3 = -3 \quad a = \sqrt{-3}$

$a = \omega + \omega + 1 \quad \omega = \dfrac{-1 \pm \sqrt{-3}}{2} = -\dfrac{1}{2} \pm i\dfrac{\sqrt{3}}{2} \quad \omega \leftrightarrow \omega^2$

$n = 7 \quad \mathrm{ord}(\omega) = 7 \quad \Phi_7(x) = \sum_0^6 x^i \quad G = \mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}) \simeq \mathbb{Z}/7^*$

$= \langle 3 \rangle \quad G = \langle \sigma_3 \rangle \quad \sigma_3 : \omega \mapsto \omega^3$

$$\begin{tikzcd}
 & K = \mathbb{Q}(\omega) \arrow[dl, dash, "3"'] \arrow[dr, dash, "2"] & \\
C_K(\sigma_2) \arrow[dr, dash, "2"'] & & C_K(\sigma_6) \arrow[dl, dash, "3"] \\
 & \mathbb{Q} &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & id \arrow[dl, dash, "3"'] \arrow[dr, dash, "2"] & \\
\langle\sigma_2\rangle \arrow[dr, dash, "2"'] & & \langle\sigma_6\rangle \arrow[dl, dash, "3"] \\
 & \langle\sigma_3\rangle \simeq C_6 &
\end{tikzcd}
\qquad \sigma_3^3$$

$\sigma_2 : \omega \mapsto \omega^2 \qquad C_K(\sigma_2)?$

$\eta = \omega + \omega^2 + \omega^4 \in C_K(\sigma_2) \quad \sigma_3(\eta) = \omega^3 + \omega^6 + \omega^5 \neq \eta$

altrimenti $1 + \omega + \omega^3 - \omega^2 - \omega^5 - \omega^4 = 0$ ossia

ω annullerebbe polinomio di grado 5

$\min_{\mathbb{Q}}(\eta)? \qquad \min_{\mathbb{Q}}(\eta) = \prod_{\beta \in \eta^G} (x - \beta)$

dove $\eta^G = \{\gamma(\eta) : \gamma \in G\} \qquad |\eta^G| = |G : C_G(\eta)| = 2$

$(x - \eta)(x - \sigma_3(\eta)) = x^2 - (\eta + \sigma_3(\eta))x + \eta\sigma_3(\eta)$

$\eta + \sigma(\eta) = \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 = -1$

$2 = \eta\sigma(\eta) \underset{exp}{\rightsquigarrow} \underline{4}, \underline{0}, \underline{6}, \underline{5}, \underline{1}, \underbrace{0, 0}, \underline{3}, \underline{2}$

$x^2 + x + 2 \qquad \dfrac{-1 \pm \sqrt{-7}}{2} \qquad C_K(\sigma_2) = \mathbb{Q}[\sqrt{-7}] = \mathbb{Q}[\eta]$

Ex

$$\begin{tikzcd} K \arrow[d, dash] \\ L \end{tikzcd} \qquad \exists\, a \in K$$

$L = \mathbb{Q}[\sqrt{-7}]$

i) $K = L[a]$

ii) $a^3 \in L$

Se $a^3 = b \in L$ $\quad \min_L(a) = x^3 - b$

Sia a' altra radice di $x^3 - b$ $\quad a' \in K$ $\quad K/\mathbb{Q}$ normale

$a'/a = \lambda$? $\quad (a'/a)^3 = \dfrac{(a')^3}{a^3} = \dfrac{b}{b} = 1 = \lambda^3$ $\quad \lambda \neq 1$ $\quad \mathrm{ord}(\lambda) = 3$

quindi $\lambda \in K$ Assurdo Infatti se $\lambda \in K$

$\mathrm{ord}(\omega) = 7$, $\mathrm{ord}(\lambda) = 3$ $\quad \omega, \lambda \in K$ $\quad \omega\lambda \in K$

e ha ordine 21 $\quad \mathbb{Q}(\omega\lambda) \leq K$

$$\begin{tikzcd} \mathbb{Q}(\omega\lambda) \arrow[d, dash, "12 = \varphi(21)"'] & K \arrow[d, dash, "6"] \\ \mathbb{Q} & \mathbb{Q} \end{tikzcd}$$

Assurdo

Il motivo per cui non ottengo a : $a^{|K:L|} \in L$

è la mancanza di radici di ordine $|K:L|$

Aggiungo a K le radici cubiche unità $T = \mathbb{Q}(\lambda)$

$\mathrm{ord}(\lambda) = 3$

$$\begin{tikzcd} K \arrow[r, dash] \arrow[d, dash] & \underset{\approx}{KT} \arrow[d, dash] \\ \mathbb{Q} \arrow[r, dash] & T \end{tikzcd}$$

Teo: (Aggiunzione delle irrazionalità)

$$\begin{tikzcd} K \arrow[r, dash] \arrow[d, dash] & KT \arrow[d, dash] \\ F \arrow[r, dash] & T \end{tikzcd}$$

$K \geq F$ finita Galois, $F \leq T$

i) $KT \geq T$ è di Galois

ii) $\mathrm{Gal}(KT/T) \simeq \mathrm{Gal}(K/K\cap T) \leq \mathrm{Gal}(K/F)$

$\left(KT/T \simeq K/K\cap T \right)$ $\quad \left(\dfrac{U+W}{W} \simeq \dfrac{U}{U \cap W} \right)$

$K = \mathbb{Q}(\omega)$ $\quad T = \mathbb{Q}(\lambda)$

$$\begin{tikzcd} LT \arrow[r, dash, "3"] \arrow[d, dash, "{\sqrt{-7},\sqrt{-3}}"] & KT \\ T \arrow[d, dash, "\sqrt{-3}"] & \\ \mathbb{Q} & \end{tikzcd}$$

KT/T Galois $\quad KT/_{LT}$ Galois ha grado 3 $K/_{L}$

$KT \ni a$ tale che i) $KT = LT[a]$

ii) $a^3 = b \in LT$

$\beta \in C_K(\sigma_6) \quad \beta = \omega + \omega^6$ ha grado 3 su $\mathbb{Q}$.

Notate che β^G ha 3 coniugati $\beta, \sigma_3(\beta), \sigma_3^2(\beta)$

$\tau = \sigma_3 \qquad \beta, \tau(\beta), \tau^2(\beta) \qquad \tau$ si estende a KT

$\tau(\lambda) = \lambda$

$$a = \beta + \lambda\tau(\beta) + \lambda^2\tau^2(\beta)$$

$$\tau(a) = \tau(\beta) + \lambda\tau^2(\beta) + \lambda^2\beta = \lambda^2 a$$

$$\tau(a^3) = a^3 \in C_{KT}(\tau) = LT$$

MAGMA $\quad a^3 = 21\lambda + 14 \qquad a = \sqrt[3]{21\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) + 14}$

$$\mathbb{Q}(a, \sqrt{-7}, \sqrt{-3}) \ni \omega$$