Existence and uniqueness proof for John Thompson's sporadic simple group Th

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LAST BUT NOT LEAST?

Centralizer of an involution

In 1976 Thompson announced the existence of a new simple group T living in $\mathrm{E}_8(3)$. It satisfies

- T has only one class of involutions;
- If $H = C_T(z)$, z an involution, then $\mathrm{O}(H)$ is extraspecial of order 2^9 and $H/\mathrm{O}(H) \cong Alt_9$.

In 1977 Parrott proved

Teorema 1. *Let G be a finite group, $z \in G$ an involution, and $H = C_G(z)$. Assume that*

- $G \neq C_G(z)\mathrm{O}(G)$;

- $\mathrm{O}(H)$ *is extraspecial of order* 2^9;

- $H/\mathrm{O}(H) \cong Alt_9$.

then G *is a simple group with the same order and conjugacy classes as Thompson's simple group* T.

Parrott also proved that $\mathrm{O}(H)$ is 2_+^{1+8}, namely the central product of dihedral groups, and H is a non-split extension.

Using his own algorithm `ExtensionsOfSolubleGroup` implemented in MAGMA, Holt has built two non-isomorphic non-split extensions of 2_+^{1+8} by Alt_9.

Michler's Algorithm

Our starting point will be the group $H \cong 2_+^{1+8}.Alt_9$ as given in the ATLAS or in a paper by Havas, Soicher, and Wilson.

We say that G is a group of **Thompson-type** if there exists an involution $z \in Z(S)$, $S \in \mathrm{Syl}_2(G)$, such that $C_G(z) \cong H$.

There exists $A \trianglelefteq S \in \mathrm{Syl}_2(H)$ elementary abelian of rank 5 such that $D = N_H(A) < H$ is maximal. Moreover D/A is maximal parabolic subgroup of $\mathrm{GL}_5(2)$ and the extension is non-split. Demwolff has proven that there exists a unique non-split extension E of 2^5 by $\mathrm{GL}_5(2)$.

Applying Michler's algorithm we have that $E \cong N_G(A)$, for any group G of Thompson type such that $G \neq H\mathrm{O}(G)$. An easy fusion argument shows that G has one conjugacy class of involutions.

Generating with 2-local subgroups

Since S is not semidihedral or dihedral, by Bender-Suzuki Theorem G has no **strongly embedded subgroup**. So we apply

Teorema 2. *Assume G has no strongly embedded subgroups. Let $H = C_G(z)$, z a 2-central involution, $S \in \mathrm{Syl}_2(G)$, and $I(H)$ be the set of involutions t of H. Then*

$$G = \langle H, N_G(S), C_G(t) \mid t \in I(H) \rangle.$$

and obtain that any group of Thompson type is generated by H and the Dempwolff group E.

More precisely G is an epimorphic image of the **amalgamated product** $H *_D E$. Now $\mathrm{Aut}(D) = H^* E^*$, where $L^* = \{\alpha_{\restriction D} : \alpha \in N_{\mathrm{Aut}(L)}(D)\}$.

Building a group of Thompson type

Weller, using $G = \langle H, E\rangle$ has built a group of Thompson type determining a compatible pair of degree 248 over $\mathbb{F}_{11}$. Applying his and Michler's algorithm to a permutation representation of degree $143,127,000$ with stabilizer ${}^3\mathrm{D}_4(3) : 3$, he has built the character table and the conjugacy classes of this group and proved it admits a unique class function χ of degree 248. Applying **Brauer's characterization of characters** he proves $\chi \in \mathrm{Irr}(G)$, showing $\chi_{\restriction N} \in Ch(N)$ for any $N = N_G(g)$, g a p-central element and $(\chi, \chi)_G = 1$.

How many choices?

Teorema 3 (Goldschmidt). *The number of isomorphism classes of amalgamated products* $H *_D E$ *equals the number of double* (H^*, E^*)*-cosets in* $\mathtt{Aut}(D)$*.*

We call this number the **Goldschmidt index** of the amalgamated product.

According to Michler's algorithm we seek multiplicity-free compatible pairs, namely characters η of H and ε of E such that $\eta_{\restriction D} = \varepsilon_{\restriction D} = \delta$.

We need to count how many characters μ of $H *_D E$ satisfy

$$\mu_{\restriction H} = \eta, \quad \mu_{\restriction E} = \varepsilon.$$

Counting modules

Teorema 4 (Thompson). *Let H, E be finite groups, $D = H \cap E$, F be a finite field of characteristic not dividing $|D|$. Let V and W be an FH and FE-module such that $V_{\upharpoonright D} \cong W_{\upharpoonright D}$. Set $L = \mathsf{GL}_{\mathsf{dim}(V)}(F)$. Then the number of $F[H *_D E]$-modules M such that $M_{\upharpoonright H} \cong V$ and $M_{\upharpoonright E} \cong W$ equals the number of $(C_L(H), C_L(E))$ double cosets of $C_L(D)$.*

Since δ is multiplicity-free, Schur's Lemma proves that the number of double cosets above equals $(|F| - 1)^{m+1-k-l}$, where η, ε, and δ have k, l, and m irreducible constituents. In our situation $k = 2$, $l = 1$ and $m = 2$.

Michler's Uniqueness Criterion

Teorema 5. *Assume G satisfy*

1. *there exists a unique class g^G of p-central elements of order a prime p;*

2. *there exists a non-cyclic elementary abelian normal subgroup A of $P \in \mathsf{Syl}_p(H)$, $H = N_G(g)$, with $D = N_H(A) < H$;*

3. $G = \langle N, E \rangle$, $E = N_G(A)$;

4. *the Goldschmidt index for (H, E, D) is 1;*

5. *there exists a unique up to duality representation $\kappa : H *_D E \to \mathsf{GL}_n(F)$.*

Then G is unique among groups of H-type if all such groups satisfy

a. *they have a unique class of p-central elements of order a prime p;*

b. *they have the same order as G;*

c. *they admit an irreducible representation of dimension n;*

d. *same local structure as G in 2.*

Michler's Order Formula

To apply Michler's uniqueness criterion we need to obtain the order of any group of Thompson type. We need an analogue of Thompson's order formula for groups with **only** one conjugacy class of involutions.

Let π be the smallest set of primes containing all prime divisors p of $|H|$ such that $C_G(g)$ is a π-subgroup of G for each π-element $1 \neq g \in G$. We call the elements in π **visible primes**.

Teorema 6 (Michler's order formula). *Suppose G has only one class z^G of involutions, $H = C_G(z)$. Let $r_1, r_2, \ldots, r_s$ be a set of representatives of the strongly real conjugacy classes r_i^G of π-elements of G. Let $b_1, b_2, \ldots, b_t$ be a set of representatives of the strongly real conjugacy classes b_j^G of π'-elements of G. Then the following statements hold:*

(a) $d(r_i) = |\{(x,y) \in z^G \times z^G | xy = r_i\}| \leq |C_G(r_i)|$ *for* $1 \leq i \leq s$;

(b) $d(b_j) = |\{(x,y) \in z^G \times z^G | xy = b_j\}| = |C_G(b_j)|$ *for* $1 \leq j \leq t$;

(c) *If* $c = 1 + \sum_{i=1}^{s} d(r_i) \frac{|H|}{|C_G(r_i)|}$, *then* $c \in \mathbb{N}$;

(d) $|G| = c|H| + t|H|^2$.

3-local structure

Here $\pi = \{2, 3, 5, 7, 13\}$. Using suggestions from Parrott paper we get the exact structure for $\mathrm{Syl}_3(G)$ proving it has order 3^{10} applying Brauer-Suzuki Theorem and Brauer-Wielandt Formula.

Using the previous theorem, a theorem of Frobenius on the number of solutions of $x^e = 1$ in a group and an argument of Lyons we obtain

Teorema 7. *Any group of Thompson-type G admits a unique irreducible character of degree 248 and has order*

$$|G| = 90745943887872000 = 2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31.$$

In particular Th is uniquely determined by the centralizer of its involution.